Graphs of Rational Functions

What if you had a function like $y = \frac{x+1}{x^2-4}$? How could you graph it and find its asymptotes? After completing this Concept, you'll be able to graph rational functions like this one.

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CK-12 Foundation: 1202S RGraphs of Rational Functions

Guidance

Graphs of rational functions are very distinctive, because they get closer and closer to certain values but never reach those values. This behavior is called asymptotic behavior, and we will see that rational functions can have **horizontal asymptotes, vertical asymptotes** or **oblique (or slant) asymptotes.**

Now we'll extend the domain and range of rational equations to include negative values of x and y . First we'll plot a few rational functions by using a table of values, and then we'll talk about the distinguishing characteristics of rational functions that can help us make better graphs.

As we graph rational functions, we need to always pay attention to values of x that will cause us to divide by 0. Remember that dividing by 0 doesn't give us an actual number as a result.

Example A

Graph the function $y = \frac{1}{x}$.

Solution

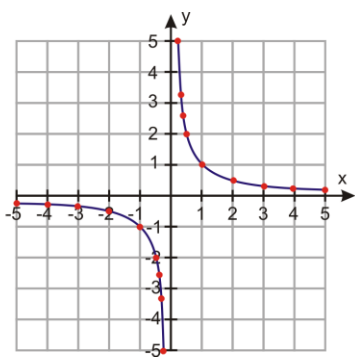
Before we make a table of values, we should notice that the function is not defined for $x = 0$. This means that the graph of the function won't have a value at that point. Since the value of $x = 0$is special, we should make sure to pick enough values close to $x = 0$ in order to get a good idea how the graph behaves.

Let's make two tables: one for $x-$ values smaller than zero and one for $x-$ values larger than zero.

x	$y = \frac{1}{x}$	x	$y = \frac{1}{x}$

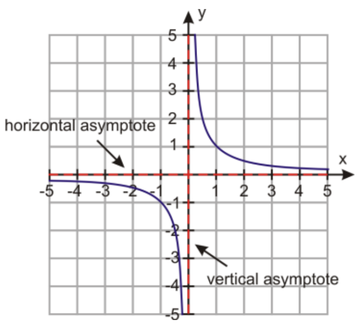
x	$y=\frac{1}{x}$	x	$y=\frac{1}{x}$
-5	$y=\frac{1}{-5}=-0.2$	0.1	$y=\frac{1}{0.1}=10$
-4	$y=\frac{1}{-4}=-0.25$	0.2	$y=\frac{1}{0.2}=5$
-3	$y=\frac{1}{-3}=-0.33$	0.3	$y=\frac{1}{0.3}=3.3$
-2	$y=\frac{1}{-2}=-0.5$	0.4	$y=\frac{1}{0.4}=2.5$
-1	$y=\frac{1}{-1}=-1$	0.5	$y=\frac{1}{0.5}=2$
-0.5	$y=\frac{1}{-0.5}=-2$	1	$y=\frac{1}{1}=1$
-0.4	$y=\frac{1}{-0.4}=-2.5$	2	$y=\frac{1}{2}=0.5$
-0.3	$y=\frac{1}{-0.3}=-3.3$	3	$y=\frac{1}{3}=0.33$
-0.2	$y=\frac{1}{-0.2}=-5$	4	$y=\frac{1}{4}=0.25$
-0.1	$y=\frac{1}{-0.1}=-10$	5	$y=\frac{1}{5}=0.2$

We can see that as we pick positive values of x closer and closer to zero, y gets larger, and as we pick negative values of x closer and closer to zero, y gets smaller (or more and more negative).



Notice on the graph that for values of x near 0, the points on the graph get closer and closer to the vertical line $x = 0$. The line $x = 0$ is called a **vertical asymptote** of the function $y = \frac{1}{x}$.

We also notice that as the absolute values of x get larger in the positive direction or in the negative direction, the value of y gets closer and closer to $y = 0$ but will never gain that value. Since $y = \frac{1}{x}$, we can see that there are no values of x that will give us the value $y = 0$. The horizontal line $y = 0$ is called a **horizontal asymptote** of the function $y = \frac{1}{x}$



Asymptotes are usually denoted as dashed lines on a graph. They are not part of the function; instead, they show values that the function approaches, but never gets to. A horizontal asymptote shows the value of y that the function approaches (but never reaches) as the absolute value of $x$gets larger and larger. A vertical asymptote shows that the absolute value of y gets larger and larger as x gets closer to a certain value which it can never actually reach.

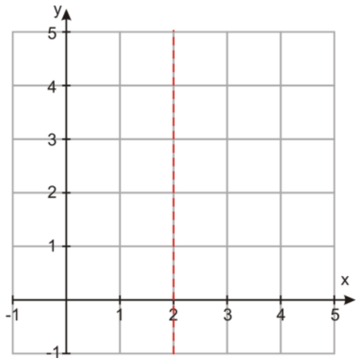
Now we'll show the graph of a rational function that has a vertical asymptote at a non-zero value of x .

Example B

Graph the function $y = \frac{1}{(x-2)^2}$ *.*

Solution

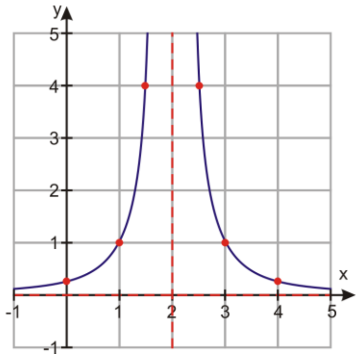
We can see that the function is not defined for $x = 2$, because that would make the denominator of the fraction equal zero. This tells us that there should be a vertical asymptote at $x = 2$, so we can start graphing the function by drawing the vertical asymptote.



Now let's make a table of values.

x	$y=\frac{1}{(x-2)^2}$
0	$y=\frac{1}{(0-2)^2}=\frac{1}{4}$
1	$y=\frac{1}{(1-2)^2}=1$
1.5	$y=\frac{1}{(1.5-2)^2}=4$
2	undefined
2.5	$y=\frac{1}{(2.5-2)^2}=4$
3	$y=\frac{1}{(3-2)^2}=1$
4	$y=\frac{1}{(4-2)^2}=\frac{1}{4}$

Here's the resulting graph:



Notice that we didn't pick as many values for our table this time, because by now we have a pretty good idea what happens near the vertical asymptote.

We also know that for large values of $|x|$, the value of y could approach a constant value. In this case that value is $y = 0$: this is the horizontal asymptote.

A rational function doesn't have to have a vertical or horizontal asymptote. The next example shows a rational function with no vertical asymptotes.

Example C

Graph the function $y = \frac{x^2}{x^2+1}$.

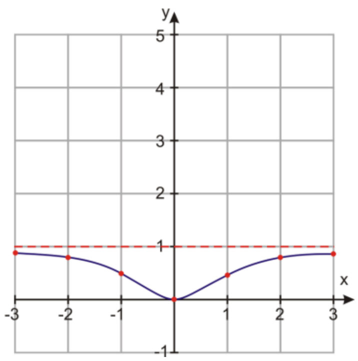
Solution

We can see that this function will have no vertical asymptotes because the denominator of the fraction will never be zero. Let's make a table of values to see if the value of y approaches a particular value for large values of x , both positive and negative.

x	$y = \frac{x^2}{x^2+1}$
-3	$y = \frac{(-3)^2}{(-3)^2+1} = \frac{9}{10} = 0.9$
-2	$y = \frac{(-2)^2}{(-2)^2+1} = \frac{4}{5} = 0.8$
-1	$y = \frac{(-1)^2}{(-1)^2+1} = \frac{1}{2} = 0.5$
0	$y = \frac{(0)^2}{(0)^2+1} = \frac{0}{1} = 0$
1	$y = \frac{(1)^2}{(1)^2+1} = \frac{1}{2} = 0.5$

x	$y = \frac{x^2}{x^2+1}$
2	$y = \frac{(2)^2}{(2)^2+1} = \frac{4}{5} = 0.8$
3	$y = \frac{(3)^2}{(3)^2+1} = \frac{9}{10} = 0.9$

Below is the graph of this function.



The function has no vertical asymptote. However, we can see that as the values of $|x|$ get larger, the value of y gets closer and closer to 1, so the function has a horizontal asymptote at $y = 1$.

Watch this video for help with the Examples above.

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Vocabulary

- Graphs of rational functions are very distinctive, because they get closer and closer to certain values but never reach those values. This behavior is called asymptotic behavior, and we will see that rational functions can have **horizontal asymptotes, vertical asymptotes** or **oblique (or slant) asymptotes.**

Guided Practice

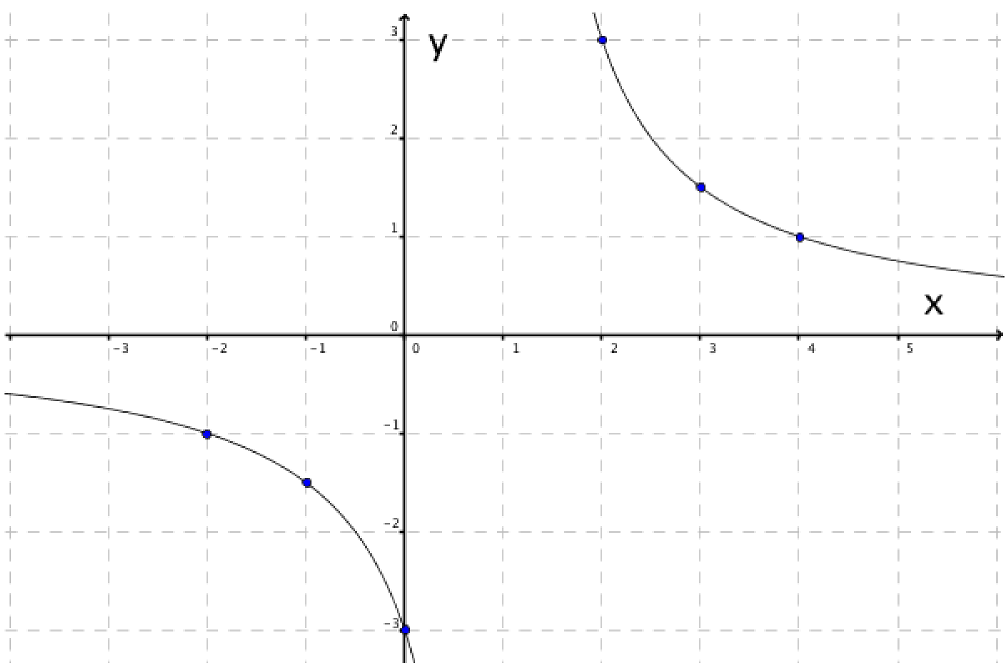
Graph the function $y = \frac{3}{x-1}$ *.*

Solution

Start by making a table of values:

x	$y = \frac{3}{x-1}$
-2	$y = \frac{3}{-2-1} = \frac{3}{-3} = -1$
-1	$y = \frac{3}{-1-1} = \frac{3}{-2} = -1.5$
0	$y = \frac{3}{0-1} = \frac{3}{-1} = -3$
1	undefined
2	$y = \frac{3}{2-1} = \frac{3}{1} = 3$
3	$y = \frac{3}{3-1} = \frac{3}{2} = 1.5$
4	$y = \frac{3}{4-1} = \frac{3}{3} = 1$

Next, graph the points. Recall that the function $y = \frac{1}{x}$ has two curves, that are on either side of the vertical asymptote, which is where the function is undefined. The same is true for this function.



Practice

Graph the following rational functions. Draw dashed vertical and horizontal lines on the graph to denote asymptotes.

1. $y = \frac{2}{x-3}$
2. $y = \frac{3}{x^2}$
3. $y = \frac{x}{x-1}$
4. $y = \frac{2x}{x+1}$
5. $y = \frac{-1}{x^2+2}$
6. $y = \frac{x}{x^2+9}$
7. $y = \frac{x^2}{x^2+1}$
8. $y = \frac{1}{x^2-1}$
9. $y = \frac{2x}{x^2-9}$
10. $y = \frac{x^2}{x^2-16}$
11. $y = \frac{3}{x^2-4x+4}$
12. $y = \frac{x}{x^2-x-6}$