Inverse Functions

Learning objectives

- Find the inverse of a function
- Determine if a function is invertible
- State the domain and range for a function and its inverse
- Graph functions and their inverses
- Use composition to verify if two functions are inverses.

Introduction

In chapters 1 and 2 we have examined properties and applications of several families of functions. In this chapter, we will focus on two related functions: exponential functions,and logarithmic functions. These two functions have a special relationship with one another: they are **inverses** of each other. In this first lesson we will develop the idea of inverses, both algebraically and graphically, as background for studying these two types of functions in depth. We will begin with a familiar, every-day example of two functions that are inverses.

Functions and inverses

In the United States, we measure temperature using the Fahrenheit scale. In other countries, people use the Celsius scale. The equation *C* = 5/9 (*F* - 32) can be used to find C, the Celsius temperature, given *F*, the Fahrenheit temperature. If we write this equation using function notation, we have $t(x) = \frac{5}{9}(x - 32)$. The input of the function is a Fahrenheit temperature, and the output is a Celsius temperature. For example, the freezing point on the Fahrenheit scale is 32 degrees. We can find the corresponding Celsius temperature using the function:

$$t(32) = \frac{5}{9}(32 - 32) = \frac{5}{9} \cdot 0 = 0$$

This function allows us to convert a Fahrenheit temperature into Celsius, but what if we want to convert from Celsius to Fahrenheit?

Consider again the equation above: $C = \frac{5}{9}(F - 32)$. We can solve this equation to isolate F:

$$C = \frac{5}{9}(F - 32)$$

$$\frac{9}{5}C = \frac{9}{5} \times \frac{5}{9}(F - 32)$$

$$C = \frac{5}{9}(F - 32)$$

$$\frac{9}{5}C = F - 32$$

$$\frac{9}{5}C + 32 = F$$

If we write this equation using function notation, we get $f(x) = \frac{9}{5}x + 32$. For this function, the input is the Celsius temperature, and the output is the Fahrenheit temperature. For example, if $x = 0, f(0) = \frac{9}{5}(0) + 32 = 0 + 32 = 32$.

Now consider the functions $t(x) = \frac{5}{9}(x - 32)$ and $f(x) = \frac{9}{5}x + 32$ together. The input of one function is the output of the other. This is an informal way of saying that these functions are **inverses.** Formally, the inverse of a function is defined as follows:

Inverse functions

Functions *f*(x) *and* g(x) *are said to inverses if*

f(g(x)) = g(f(x)) = x

Or, using the composite function notation of Chapter 1:

$f \circ g = g \circ f = x$

The following notation is used to indicate inverse functions:

If *f*(*x*) and *g*(*x*) are inverse functions, then

$f(x) = g^{-1}(x)$ and $g(x) = f^{-1}(x)$

The following notation is also used: $f = g^{-1}$ **and** $g = f^{-1}$.

Note that $f^{-1}(x)$ ***does not equal*** $\frac{1}{f(x)}$.

Informally, we define the inverse of a function as the relation we obtain by switching the domain and range of the function. Because of this definition, you can find an inverse by switching the roles of *x* and *y* in an equation. For example, consider the function *g*(*x*) = 2*x*. This is the line *y* = 2*x.* If we switch *x*and *y*, we get the equation *x* = 2*y.* Dividing both sides by 2, we get *y* = 1/2 *x*. Therefore the functions *g(x)* = 2*x* and *y* = 1/2 *x* are inverses. Using function notation, we can write *y* = 1/2 *x* as $g^{-1}(x) = 1/2\ x$.

Example 1: Find the inverse of each function

a. $f(x) = 5x - 8$	**b.** $f(x) = x^3$.

Solution:

a. First write the function using " y =" notation, then interchange *x* and *y*:

$f(x) = 5x - 8 \rightarrow y = 5x - 8 \rightarrow x = 5y - 8$

Then isolate *y*:

$$x = 5y - 8$$

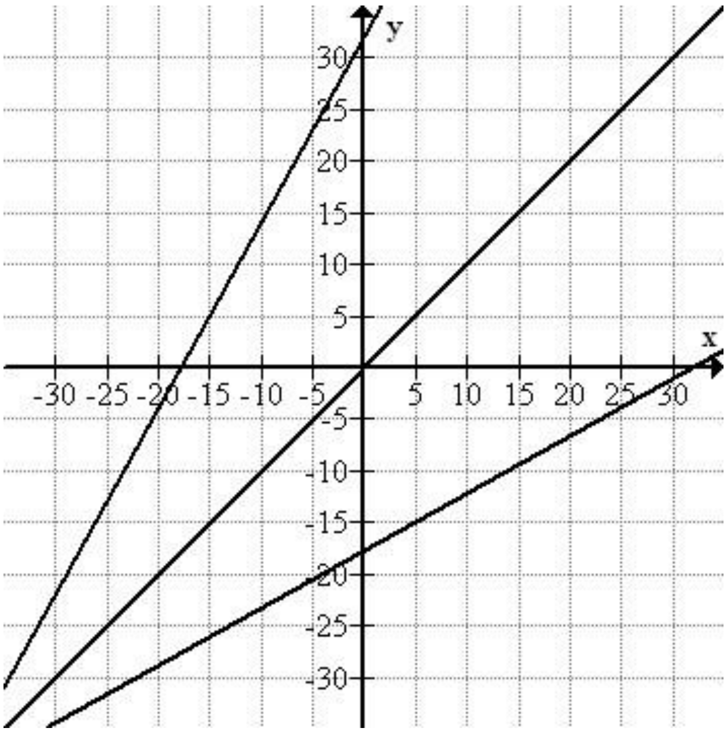
$$x + 8 = 5y$$

$$y = \frac{1}{5}x + \frac{8}{5}$$

b.

First write the function using " y= ":	$f(x) = x^3$
	$y = x^3$
Now interchange *x* and *y*	$x = y^3$
Now isolate *y*:	$y = \sqrt[3]{x}$

Because of the definition of inverse, the graphs of inverses are reflections across the line $y = x$. The graph below shows $t(x) = \frac{5}{9}(x - 32)$ and $f(x) = \frac{9}{5}x + 32$ on the same graph, along with the reflection line $y = x$.

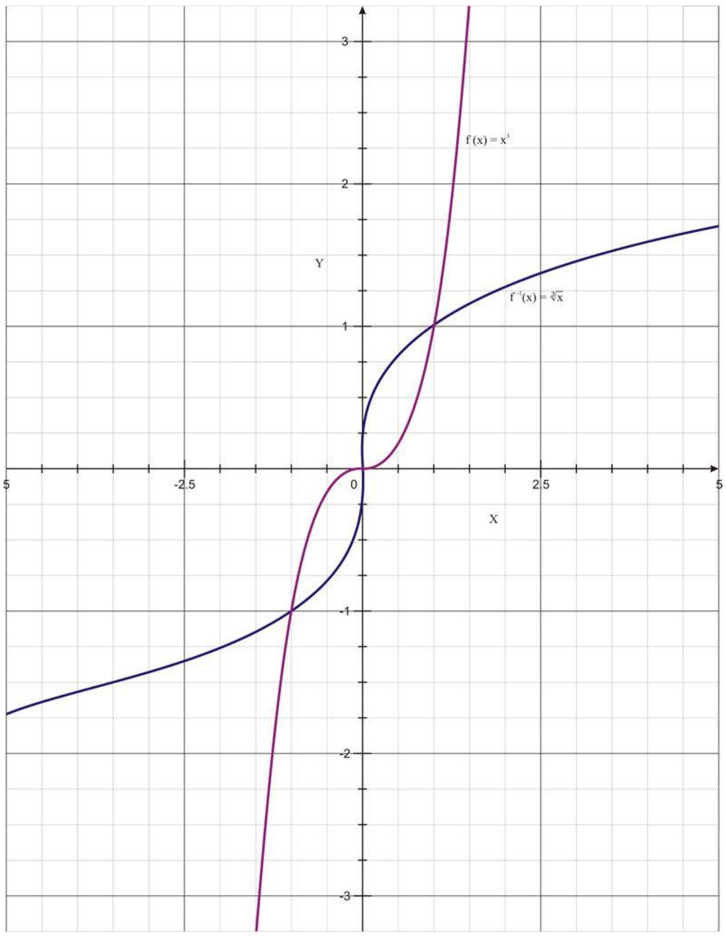


A note about graphing with software or a graphing calculator: if you look at the graph above, you can see that the lines are reflections over the line *y* = *x*. However, if you do not view the graph in a window that shows equal scales of the *x*- and *y*-axes, the graph might not look like this.

Before continuing, there are two other important things to note about inverses. First, remember that the "' -1'" is not an exponent, but a symbol that represents an inverse. Second, not every function has an inverse that is a function. In the examples we have considered so far, we inverted a function, and the resulting relation was also a function. However, some functions are not **invertible**; that is, following the process of "inverting" them does not produce a relation that is a function. We will return to this issue below when we examine domain and range of functions and their inverses. First we will look at a set of functions that *are* invertible.

Inverses of 1-to-1 functions

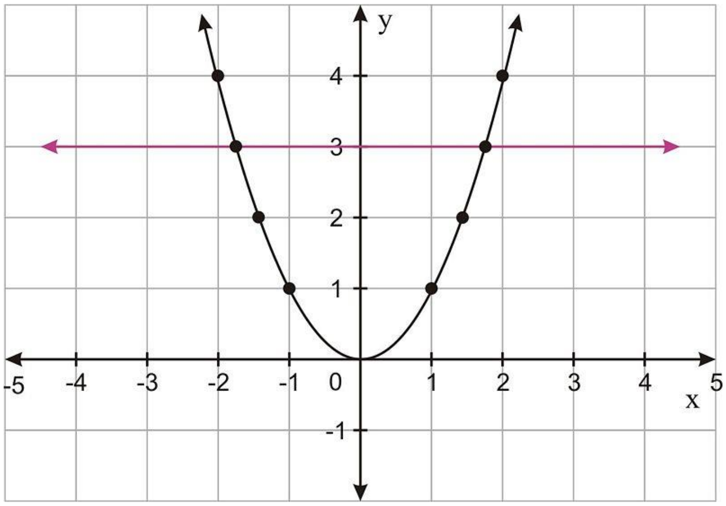
Consider again example 1 above. We began with the function $f(x) = x^3$, and we found the inverse $f^{-1}(x) = \sqrt[3]{x}$. The graphs of these functions are show below.



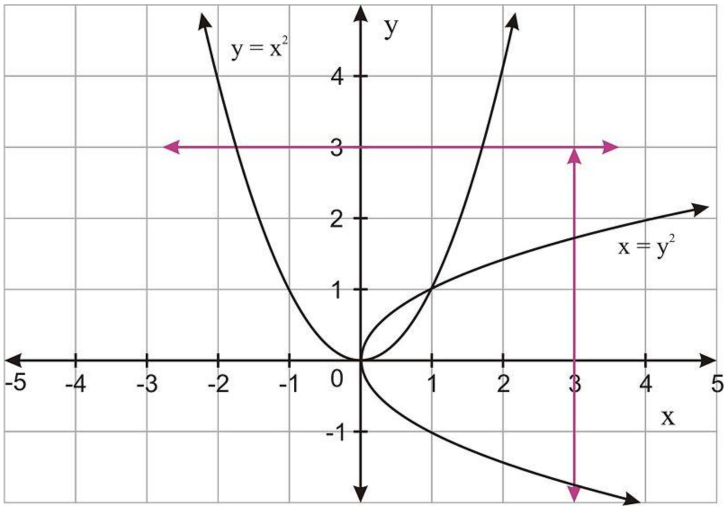
The function $f(x) = x^3$ is an example of a **one-to-one function,** which is defined as follows:

> **One to one**
> **A function is one-to-one if and only if every element of its domain corresponds to *exactly* one element of its range.**

The linear functions we examined above are also one-to-one. The function $y = x^2$, however, is not one-to-one. The graph of this function is shown below.



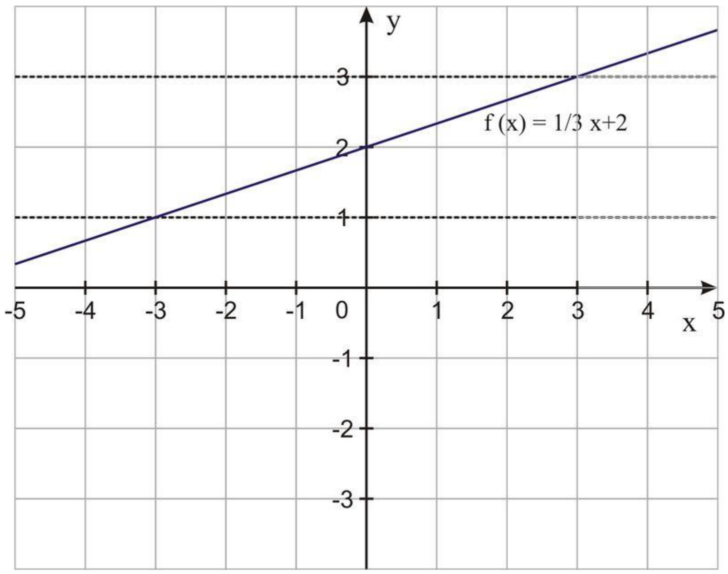
You may recall that you can identify a relation as a function if you draw a vertical line through the graph, and the line touches only one point. Notice then that if we draw a horizontal line through $y = x^2$, the line touches more than one point. Therefore if we inverted the function, the resulting graph would be a reflection over the line $y = x$, and the inverse would not be a function. It fails the vertical line test.



The function $y = x^2$ is therefore *not* a one-to-one function. A function that *is*one-to-one will be invertible. You can determine this graphically by drawing a horizontal line through the graph of the function. For example, if you draw a horizontal line through the graph of $f(x) = x^3$, the line will only touch one point on the graph, no matter where you draw the line.

Example 2: Graph the function $f(x) = \frac{1}{3}x + 2$. Use a horizontal line test to verify that the function is invertible.

Solution: The graph below shows that this function is invertible. We can draw a horizontal line at any *y* value, and the line will only cross $f(x) = \frac{1}{3}x + 2$ once.



In sum, a one-to-one function is invertible. That is, if we invert a one-to-one function, its inverse is also a function. Now that we have established what it means for a function to be invertible, we will focus on the domain and range of inverse functions.

Domain and range of functions and their inverses

Because of the definition of inverse, a function's domain is its inverse's range, and the inverse's domain is the function's range. This statement may seem confusing without a specific example.

Example 3: State the domain and range of the function and its inverse:

Function: *(1, 2), (2, 5), (3, 7)*

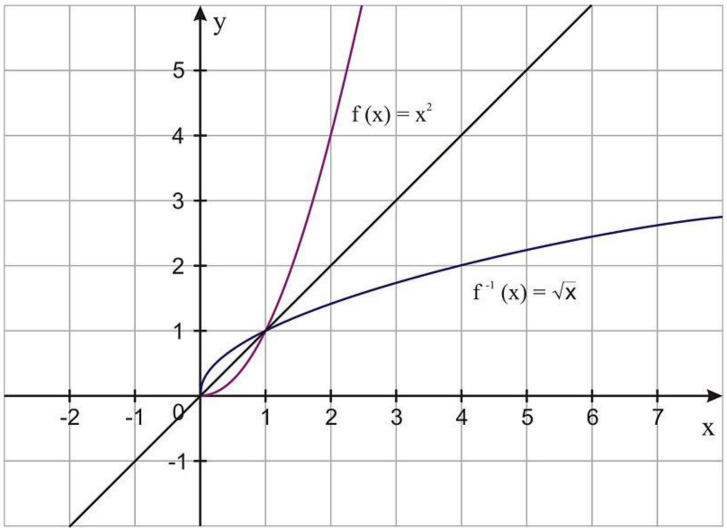
Solution: the inverse of this function is the set of points *(2, 1), (5, 2), (7, 3)*

The domain of the function is *{1, 2, 3}*. This is also the range of the inverse.

The range of the function is *{2, 5, 7}*. This is also the domain of the inverse.

The linear functions we examined previously, as well as $f(x) = x^3$, all had domain and range both equal to the set of all real numbers. Therefore the inverses also had domain and range equal to the set of all real numbers. Because the domain and range were the same for these functions, switching them maintained that relationship.

Also, as we found above, the function $y = x^2$ is not one-to-one, and hence it is not invertible. That is, if we invert it, the resulting relation is not a function. We can change this situation if we define the domain of the function in a more limited way. Let $f(x)$ be a function defined as follows: $f(x) = x^2$, with domain limited to real numbers ≥ 0. Then the inverse of the function is the square root function: $f^{-1}(x) = \sqrt{x}$



Example 4: Define the domain for the function $f(x) = (x - 2)^2$ so that f is invertible.
Solution: The graph of this function is a parabola. We need to limit the domain to one side of the parabola. Conventionally in cases like these we choose the positive side; therefore, the domain is limited to real numbers ≥ 2.

Inverse functions and composition

In the examples we have considered so far, we have taken a function and found its inverse. We can also analyze two functions and determine whether or not they are inverses. Recall the formal definition from above:

Two functions $f(x)$ and $g(x)$ are inverses if and only if $f(g(x)) = g(f(x)) = x$.

This definition is perhaps easier to understand if we look at a specific example. Let's use two functions that we have established as inverses: $f(x) = 2x$ and $g(x) = 1/2\ x$. Let's also consider a specific x value. Let $x = 8$. Then we have $f(g(8)) = f(1/2 \times 8) = f(4) = 2(4) = 8$. Similarly we could establish that $g(f(8)) = 8$. Notice that there is nothing special about $x = 8$. For any x value we input into$f$, the same value will be output by the composed functions:

$$f(g(x)) = f\left(\tfrac{1}{2}x\right) = 2\left(\tfrac{1}{2}x\right) = x$$

$$g(f(x)) = g(2x) = \tfrac{1}{2}(2x) = x$$

Example 5: Use composition of functions to determine if $f(x) = 2x + 3$ and $g(x) = 3x - 2$ are inverses.
Solution: The functions are not inverses.

We only need to check one of the compositions:

f(g(x)) = f(3x - 2) = 2(3x - 2) + 3 = 6x - 4 + 3 = 6x - 1 ≠ x

Lesson Summary

In this lesson we have defined the concept of inverse, and we have examined functions and their inverses, both algebraically and graphically. We established that functions that are one-to-one are invertible, while other functions are not necessarily invertible. (However, we can redefine the domain of a function such that it is invertible.) In the remainder of the chapter we will examine two families of functions whose members are inverses.

Points to Consider

1. Can a function be its own inverse? If so, how?
2. Consider the other function families you learned about in chapter 1. What do their inverses look like?
3. How is the rate of change of a function related to the rate of change of the function's inverse?

Review Questions

1. Find the inverse of the function $f(x) = \frac{1}{2}x - 7$.
2. Use the horizontal line test to determine if the function $f(x) = x + \frac{1}{x}$ is invertible or not.
3. Use composition of functions to determine if the functions are inverses: $g(x) = 2x - 6$ and $h(x) = \frac{1}{2}x + 3$.
4. Use composition of functions to determine if the functions are inverses: $f(x) = x + 2$ and $p(x) = x - \frac{1}{2}$.
5. Given the function $f(x) = (x + 1)^2$, how should the domain be restricted so that the function is invertible?
6. Consider the function $f(x) = \frac{3}{2}x + 4$.
   a. Find the inverse of the function.
   b. State the slope of the function and its inverse. What do you notice?
7. Given the function (0, 5), (1, 7), (2, 13), (3, 19)
   a. Find the inverse of the function.
   b. State the domain and range of the function.

c. State the domain and range of the inverse.

8. Consider the function $a(x) = |x|$

   a. Sketch the inverse of this function.

   b. Is the inverse a function? Explain.

9. Consider the function *f(x)* = *c*, where *c* is a real number. What is the inverse? Is *f* invertible? Explain.

10. A store sells fabric by the length. Red velvet goes on sale after Valentine’s day for $4.00 per foot.

    a. Write a function to model the cost of *x* feet of red velvet.

    b. What is the inverse of this function?

    c. What does the inverse represent?

Vocabulary

Inverse

The inverse of a function is the relation obtained by interchanging the domain and range of a function.

Invertible

A function is invertible if its inverse is a function.

One-to-one

A function is one-to-one if every element of its domain is paired with exactly one element of its range.