Linear and Exponential Models

Learning Objectives

- Identify functions using differences and ratios.
- Write equations for functions.
- Perform exponential and quadratic regressions with a graphing calculator.
- Solve real-world problems by comparing function models.

Introduction

In this course you have learned about three types of functions, linear, quadratic and exponential.

Linear functions take the form $y = mx = b$.

Quadratic functions take the form $y = ax^2 + bx + c$.

Exponential functions take the form $y = a \cdot b^x$.

In real-world applications, the function that describes some physical situation is not given. Finding the function is an important part of solving problems. For example, scientific data such as observations of planetary motion are often collected as a set of measurements given in a table. One job for the scientist is to figure out which function best fits the data. In this section, you will learn some methods that are used to identify which function describes the relationship between the dependent and independent variables in a problem.

Identify Functions Using Differences or Ratios.

One method for identifying functions is to look at the difference or the ratio of different values of the dependent variable.

We use differences to identify linear functions.

If the difference between values of the dependent variable is the same each time we change the independent variable by the same amount, then the function is *linear.*

Example 1

Determine if the function represented by the following table of values is linear.

x	y	difference of $y-$ values
-2	-4	
		-1 + 4 = 3
-1	-1	
		2 + 1 = 3
0	2	
		5 − 2 = 3
1	5	
		8 − 5 = 3
2	8	

If we take the difference between consecutive $y-$ values, we see that each time the $x-$ value increases by one, the $y-$ value always increases by 3.

Since the difference is always the same, **the function is linear.**

When we look at the difference of the $y-$ values, we must make sure that we examine entries for which the $x-$ values increase by the same amount.

For example, examine the values in the following table.

difference of $x-$ values	x	y	difference of $y-$ values
	0	5	
1 - 0 = 1			-1 + 4 = 3
	1	10	
3 - 1 = 2			2 + 1 = 3
	3	20	
4 - 3 = 1			5 − 2 = 3
	4	25	
6 - 4 = 2			8 − 5 = 3
	6	35	

At first glance, this function might not look linear because the difference in the $y-$ values is not always the same.

However, we see that the difference in $y-$ values is 5 when we increase the $x-$ values by 1, and it i 10 when we increase the $x-$ values by 2. This means that the difference in $y-$ values is always 5 when we increase the $x-$ values by 1. Therefore, the function is linear. The key to this observation is that **the ratio of the differences is constant** .

In mathematical notation, we can write the linear property as follows.

If $\frac{y_2 - y_1}{x_2 - x_1}$ is always the same for values of the dependent and independent variables, then the points are on a line. Notice that the expression we wrote is the definition of the slope of a line.

Differences can also be used to identify quadratic functions. For a quadratic function, when we increase the $x-$ values by the same amount,

the difference between $y-$ values will not be the same. However, the difference of the differences of the $y-$ values will be the same.

Here are some examples of quadratic relationships represented by tables of values.

a)

x	$y = x^2$	difference of $y-$ values	difference of differences
0	0	1 – 0 = **1**	
1	1	4 – 1 = **3**	3 – 1 = **2**
2	4	9 – 4 = **5**	5 – 3 = **2**
3	9	16 – 9 = **7**	7 – 5 = **2**
4	16	25 – 16 = **9**	9 – 7 = **2**
5	25	36 – 25 = **11**	11 – 9 = **2**
6	36		

In this quadratic function, $y = x^2$, when we increase the $x-$ value by one, the value of $y$increases by different values. However, the increase is constant: the difference of the difference is always 2.

b)

x	$y = 2x^2$ 3x + 1	difference of $y-$ values	difference of differences
0	0	0 – 1 = **-1**	
1	1	3 – 0 = **3**	3 + 1 = **4**
2	3	10 – 3 = **7**	7 – 3 = **4**
3	10	21 – 10 = **11**	11 – 7 = **4**
4	21	36 – 21 = **15**	15 – 11 = **4**
5	36	55 – 36 = **19**	19 – 15 = **4**
6	55		

In this quadratic function, $y = x^2 - 3x + 1$, when we increase the $x-$ value by one, the value of$y$ increases by different values. However, the increase is constant: the difference of the difference is always 4.

We use ratios to identify exponential functions.

If the ratio between values of the dependent variable is the same each time we change the independent variable by the same amount, then the function is *exponential.*

Example 2

Determine if the function represented by the following table of values is exponential.

a)

x	y	ratio of y - values
0	4	
		$\frac{12}{4} = 3$
1	12	
		$\frac{36}{12} = 3$
2	36	
		$\frac{108}{36} = 3$
3	108	
		$\frac{324}{108} = 3$
4	324	

If we take the ratio of consecutive $y-$ values, we see that each time the $x-$ value increases by one, the $y-$ value is multiplied by 3.

Since the ratio is always the same, **the function is exponential.**

b)

x	y	ratio of y - values
0	240	
		$\frac{120}{240} = \frac{1}{2}$
1	120	
		$\frac{60}{120} = \frac{1}{2}$
2	60	
		$\frac{30}{60} = \frac{1}{2}$
3	30	
		$\frac{15}{30} = \frac{1}{2}$
4	15	

If we take the ratio of consecutive $y-$ values, we see that each time the $x-$ value increases by one, the $y-$ value is multiplied by $\frac{1}{2}$.

Since the ratio is always the same, **the function is exponential.**

Write Equations for Functions.

Once we identify which type of function fits the given values, we can write an equation for the function by starting with the general form for that type of function.

Example 3

Determine what type of function represents the values in the following table.

x	y
0	3

x	y
1	1
2	-3
3	-7
4	-11

Solution

Let’s first check the difference of consecutive values of y .

x	y	difference of y - values
0	5	
1	1	1 - 5 = **-4**
2	-3	-3 - 1 = **-4**
3	-7	-7+3 = **-4**
4	-11	-11+7= **-4**

If we take the difference between consecutive $y-$ values, we see that each time the $x-$ value increases by one, the $y-$ value always decreases by 4. Since the difference is always the same,**the function is linear.**

To find the equation for the function that represents these values, we start with the general form of a linear function.

$$y = mx + b$$

Here m is the slope of the line and is defined as the quantity by which y increases every time the value of x increases by one. The constant b is the value of the function when $x = 0$. Therefore, the function is

$$y = -4x + 5$$

Example 4

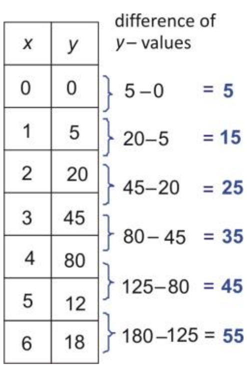
Determine what type of function represents the values in the following table.

x	y
0	0

x	y
1	5
2	20
3	45
4	80
5	125
6	180

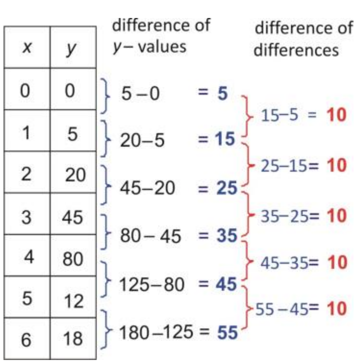
Solution

Let's first check the difference of consecutive values of y .



If we take the difference between consecutive $y-$ values, we see that each time the $x-$ value increases by one, the $y-$ value does not remain constant. Since the difference is not the same,**the function is not linear.**

Now, let's check the difference of the differences in the values of y .



When we increase the $x-$ value by one, the value of y increases by different values. However, the increase is constant. The difference of the differences is always 10 when we increase the $x-$value by one.

The function describing these set of values is **quadratic.** To find the equation for the function that represents these values, we start with the general form of a quadratic function.

$$y = ax^2 + bx + c$$

We need to use the values in the table to find the values of the constants a, b and c .

The value of c represents the value of the function when $x = 0$, so $c = 0$.

$$\begin{aligned}
\text{Then} \quad & y = ax^2 + bx \\
\text{Plug in the point } (1,5). \quad & 5 = a + b \\
\text{Plug in the point } (2,20). \quad & 20 = 4a + 2b \Rightarrow 10 = 2a + b \\
\text{To find } a \text{ and } b, \text{ we solve the system of equations} \quad & 5 = a + b \\
& 10 = 2a + b \\
\text{Solve the first equation for } b. \quad & 5 = a + b \Rightarrow b = 5 - a \\
\text{Plug the first equation into the second.} \quad & 10 = 2a + 5 - a \\
\text{Solve for a and } b. \quad & a = 5 \text{ and } b = 0
\end{aligned}$$

Therefore the equation of the quadratic function is

$$y = 5x^2$$

Example 5

Determine what type of function represents the values in the following table.

x	y
0	400
1	100
2	25
3	625
4	1.5625

Solution:

Let's check the ratio of consecutive values of y .

x	y	ratio of y - values
0	400	
1	100	$\frac{100}{400} = \frac{1}{4}$
2	25	$\frac{25}{100} = \frac{1}{4}$
3	6.25	$\frac{6.25}{25} = \frac{1}{4}$
4	1.5625	$\frac{1.5625}{6.25} = \frac{1}{4}$

If we take the ratio of consecutive $y-$ values, we see that each time the $x-$ value increases by one, the $y-$ value is multiplied by $\frac{1}{4}$.

Since the ratio is always the same, **the function is exponential.**

To find the equation for the function that represents these values, we start with the general form of an exponential function.

$$y = a \cdot b^x$$

b is the ratio between the values of y each time that x is increased by one. The constant a is the value of the function when $x = 0$. Therefore, our answer is

$$y = 400 \left(\frac{1}{4}\right)^x$$

Perform Exponential and Quadratic Regressions with a Graphing Calculator.

Earlier you learned how to perform linear regression with a graphing calculator to find the equation of a straight line that fits a linear data set. In this section, you will learn how to perform exponential and quadratic regression to find equations for functions that describe non-linear relationships between the variables in a problem.

Example 6

Find the quadratic function that is a best fit for the data in the following table. The following table shows how many miles per gallon a car gets at different speeds.

Speed (mi/h)	Miles Per Gallon
30	18

Speed (mi/h)	Miles Per Gallon
35	20
40	23
45	25
50	28
55	30
60	29
65	25
70	25

Using a graphing calculator.

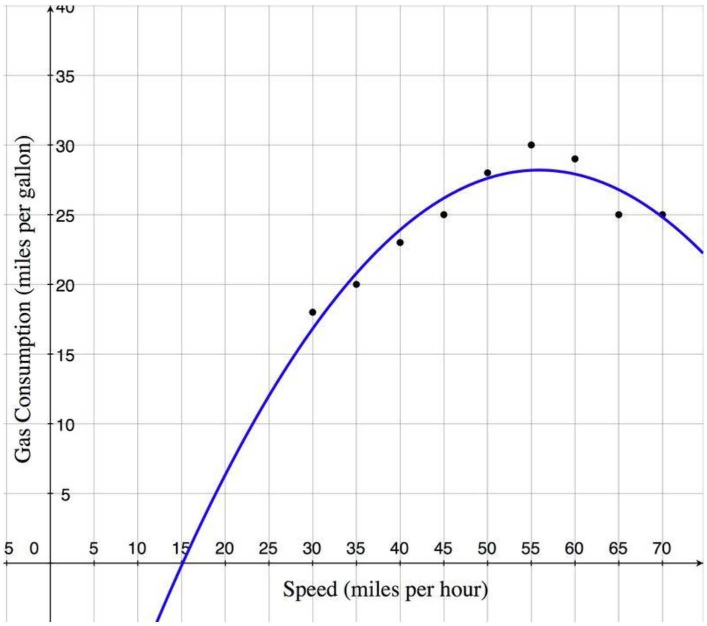
a) Draw the scatterplot of the data.

b) Find the quadratic function of best fit.

c) Draw the quadratic function of best fit on the scatterplot.

d) Find the speed that maximizes the miles per gallon.

e) Predict the miles per gallon of the car if you drive at a speed of 48 miles per gallon.



Solution

Step 1 Input the data

Press **[STAT]** and choose the **[EDIT]** option.

Input the values of x in the first column (L_1) and the values of y in the second column (L_2) .

Note: In order to clear a list, move the cursor to the top so that L_1 or L_2 is highlighted. Then press**[CLEAR]** button and then **[ENTER]** .

Step 2 Draw the scatter plot.

First press **[Y=]** and clear any function on the screen by pressing **[CLEAR]** when the old function is highlighted.

Press **[STATPLOT] [STAT]** and **[Y=]** and choose option 1.

Choose the ON option, after TYPE, choose the first graph type (scatterplot) and make sure that the Xlist and Ylist names match the names on top of the columns in the input table.

Press **[GRAPH]** and make sure that the window is set so you see all the points in the scatterplot. In this case $30 \leq x \leq 80$ and $0 \leq y \leq 40$.

You can set the window size by pressing on the **[WINDOW]** key at top.

Step 3 Perform quadratic regression.

Press **[STAT]** and use right arrow to choose **[CALC]** .

Choose Option 5 (QuadReg) and press **[ENTER]** . You will see “QuadReg” on the screen.

Type in L_1, L_2 after 'QuadReg' and Press **[ENTER]** . The calculator shows the quadratic function.

Function $y = -0.017x^2 + 1.9x - 25$

Step 4: Graph the function.

Press **[Y=]** and input the function you just found.

Press **[GRAPH]** and you will see the curve fit drawn over the data points.

To find the speed that maximizes the miles per gallons, use **[TRACE]** and move the cursor to the top of the parabola. You can also use **[CALC] [2nd] [TRACE]** and option 4 Maximum, for a more accurate answer. The speed that maximizes miles per gallons $= 56\ mi/h$

Plug $x = 56$ into the equation you found: $y = -0.017(56)^2 = 1.9(56) - 25 = 28$ miles per gallon

Note: The image to the right shows our data points from the table and the function plotted on the same graph. One thing that is clear from this graph is that predictions made with this function will not make sense for all values of x . For example, if $x < 15$, this graph predicts that we will get negative mileage, something that is impossible. Thus, part of the skill of using regression on your calculator is being aware of the strengths and limitations of this method of fitting functions to data.

Example 7

The following data represents the amount of money an investor has in an account each year for 10 years.

year	value of account
1996	$5000
1997	$5400
1998	$5800
1999	$6300
2000	$6800
2001	$7300

year	value of account
2002	$7900
2003	$8600
2004	$9300
2005	$10000
2006	$11000

Using a graphing calculator

a) Draw a scatterplot of the value of the account as the dependent variable, and the number of years *since* 1996 as the independent

variable.

b) Find the exponential function that fits the data.

c) Draw the exponential function on the scatterplot.

d) What will be the value of the account in 2020?

Solution

Step 1 Input the data

Press **[STAT]** and choose the **[EDIT]** option.

Input the values of x in the first column (L_1) and the values of y in the second column (L_2) .

Step 2 Draw the scatter plot.

First press **[Y=]** and clear any function on the screen.

Press **[GRAPH]** and choose Option 1.

Choose the ON option and make sure that the Xlist and Ylist names match the names on top of the columns in the input table.

Press **[GRAPH]** make sure that the window is set so you see all the points in the scatterplot. In this case: $0 \leq x \leq 10$ and $0 \leq y \leq 11000$.

Step 3 Perform exponential regression.

Press **[STAT]** and use right arrow to choose **[CALC]** .

Choose Option 0 and press **[ENTER]** . You will see “ExpReg” on the screen.

Press **[ENTER]** . The calculator shows the exponential function.

Function $y = 4975.7(1.08)^x$

Step 4: Graph the function.

Press **[Y=]** and input the function you just found. Press **[GRAPH]** .

Substitute $x = 2020 - 1996 = 24$ into the function $y = 4975.7(1.08)^{24} = \$31551.81$.

Note: This is a curve fit. So the function above is the curve that comes closest to all the data points. It will not return y values that are exactly the same as in the data table, but they will be close. It is actually more accurate to use the curve fit values than the data points.

Solve Real-World Problems by Comparing Function Models

Example 8

The following table shows the number of students enrolled in public elementary schools in the US (source: US Census Bureau). Make a scatterplot with the number of students as the dependent variable, and the number of years since 1990 as the independent variable. Find which curve fits this data the best and predict the school enrollment in the year 2007.

Year	Number of Students (millions)
1990	26.6
1991	26.6
1992	27.1
1993	27.7
1994	28.1
1995	28.4

Year	Number of Students (millions)
1996	28.1
1997	29.1
1998	29.3
2003	32.5

Solution

We will perform linear, quadratic and exponential regression on this data set and see which function represents the values in the table the best.

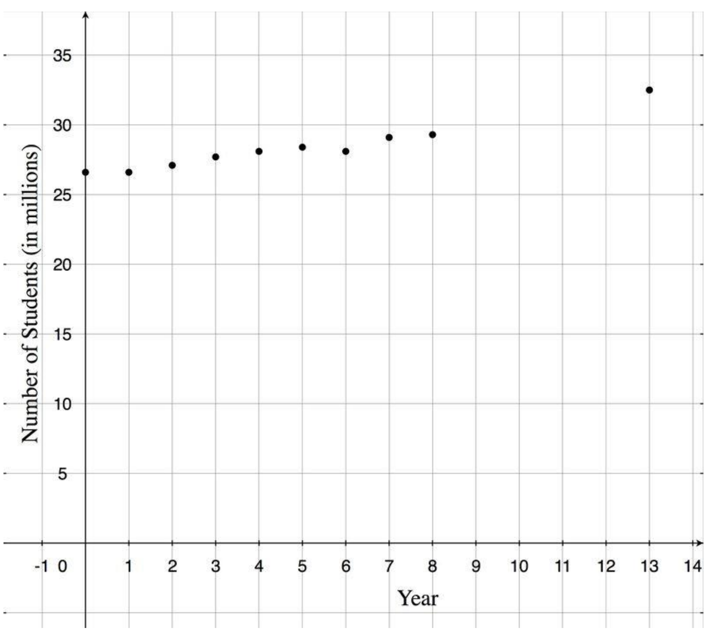
Step 1 Input the data.

Input the values of x in the first column (L_1) and the values of y in the second column (L_2).

Step 2 Draw the scatter plot.

Set the window size: $0 \leq x \leq 10$ and $20 \leq y \leq 40$.

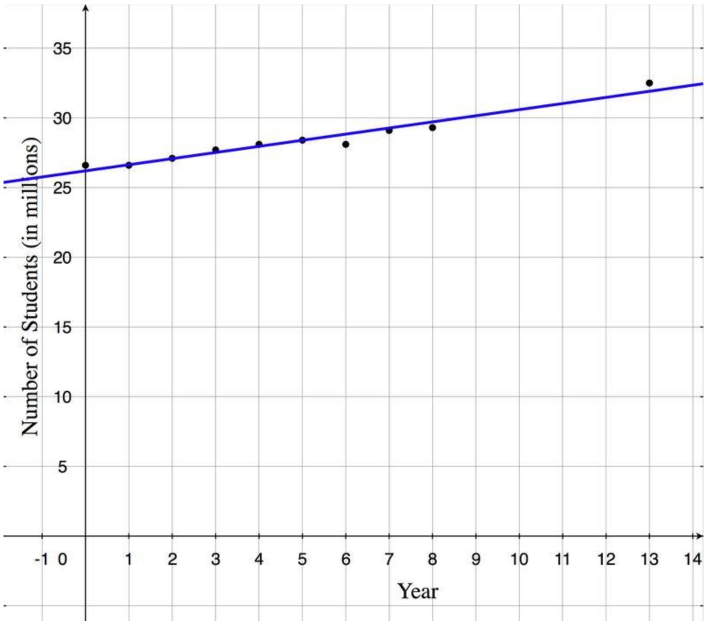
Here is the scatter plot.



Step 3 Perform Regression.

Linear Regression

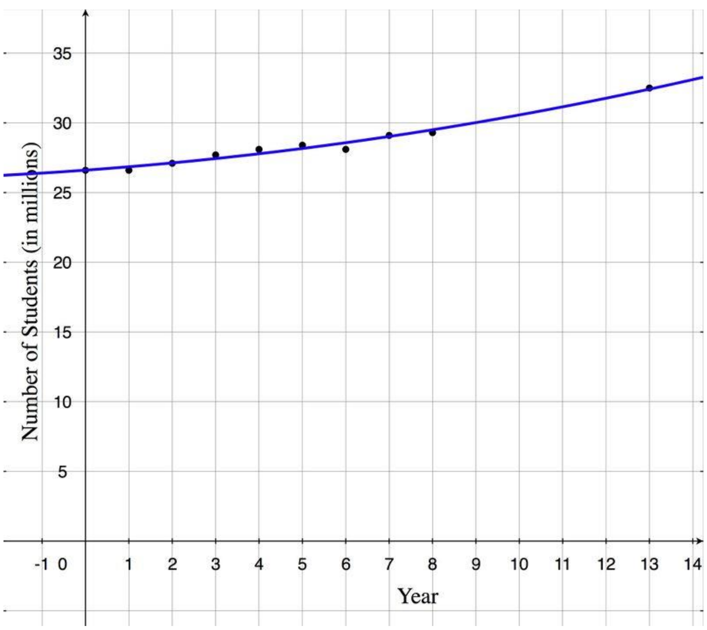
The function of the line of best fit is $y = 0.44x + 26.1$.



Here is the graph of the function on the scatter plot.

Quadratic Regression

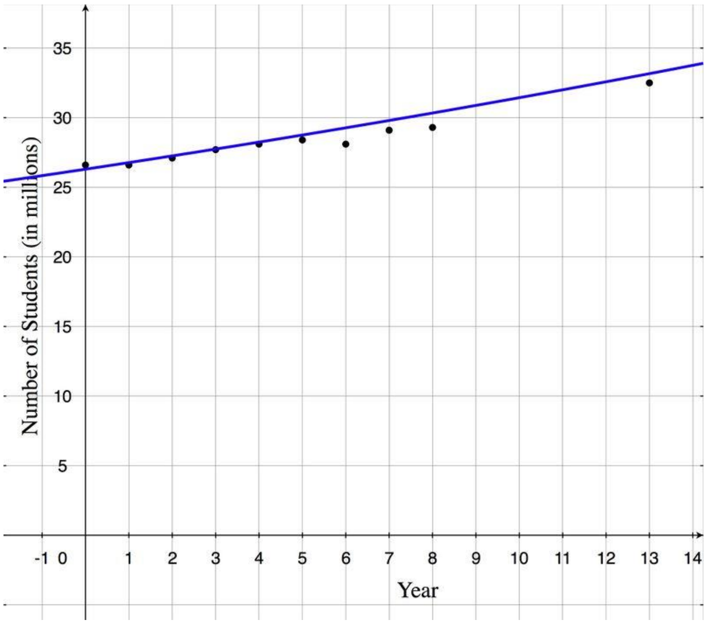
The quadratic function of best fit is $y = 0.064x^2 - .067x + 26.84$.



Here is the graph of the function on the scatter plot.

Exponential Regression

The exponential function of best fit is $y = 26.2(1.018)^x$.



Here is the graph of the function on the scatter plot.

From the graphs, it looks like the quadratic function is the best fit for this data set. We use this function to predict school enrollment in 2007.

Substitute $x = 2007 - 1990 = 17$

$y = 0.064(17)^2 - .067(17) + 26.84 = \underline{44.2 \text{ million students}}$

Review Questions

Determine whether the data in the following tables can be represented by a linear function.

x	y
-4	10
-3	7
-2	4
-1	1

x	y
0	-2
1	-5

x	y
-2	4
-1	3
0	2
1	3
2	6
3	11

x	y
0	50
1	75
2	100
3	125
4	150
5	175

Determine whether the data in the following tables can be represented by a quadratic function:

x	y
-10	10
-5	2.5
0	0
5	2.5
10	10
15	22.5

x	y
1	4
2	6
3	6
4	4
5	0
6	-6

x	y
-3	-27

x	y
-2	-8
-1	-1
0	0
1	1
2	8
3	27

Determine whether the data in the following tables can be represented by an exponential function.

x	y
0	200
1	300
2	1800
3	8300
4	25800
5	62700

x	y
0	120

x	y
1	180
2	270
3	405
4	607.5
5	911.25

x	y
0	4000
1	2400
2	1440
3	864
4	518.4
5	311.04

Determine what type of function represents the values in the following table and find the equation of the function.

x	y
0	400
1	500

x	y
2	625
3	781.25
4	976.5625

x	y
-9	-3
-7	-2
-5	-1
-3	0
-1	1
1	2

x	y
-3	14
-2	4
-1	-2
0	-4
1	-2

x	y
2	4
3	14

1. As a ball bounces up and down, the maximum height that the ball reaches continually decreases from one bounce to the next. For a given bounce, the table shows the height of the ball with respect to time.

Time (seconds)	**Height (inches)**
2	2
2.2	16
2.4	24
2.6	33
2.8	38
3.0	42
3.2	36
3.4	30
3.6	28
3.8	14
4.0	6

Using a graphing calculator

a. Draw the scatter plot of the data.
b. Find the quadratic function of best fit.

c. Draw the quadratic function of best fit on the scatter plot.
d. Find the maximum height the ball reaches on the bounce.
e. Predict how high the ball is at time $t = 2.5\ seconds$.

1. A chemist has a 250 gram sample of a radioactive material. She records the amount of radioactive material remaining in the sample every day for a week and obtains the data in the following table.

Day	**Weight(grams)**
0	250
1	208
2	158
3	130
4	102
5	80
6	65
7	50

Using a graphing calculator,

a. Draw a scatterplot of the data.
b. Find the exponential function of best fit.
c. Draw the exponential function of best fit on the scatter plot.
d. Predict the amount of material after 10 days.

1. The following table shows the rate of pregnancies (per 1000) for US women aged 15 to 19. (source: US Census Bureau). Make a scatterplot with the rate of pregnancies as the dependent variable and the number of years since 1990 as the independent variable. Find which curve fits this data the best and predict the rate of teen pregnancies in the year 2010.

Year	**Rate of Pregnancy (per 1000)**
1990	116.9

Year	Rate of Pregnancy (per 1000)
1991	115.3
1992	111.0
1993	108.0
1994	104.6
1995	99.6
1996	95.6
1997	91.4
1998	88.7
1999	85.7
2000	83.6
2001	79.5
2002	75.4