Exponential Growth Functions

Learning Objectives

- Graph an exponential growth function.
- Compare graphs of exponential growth functions.
- Solve real-world problems involving exponential growth.

Introduction

Exponential functions are different than other functions you have seen before because now the variable appears as the exponent (or power) instead of the base. In this section, we will be working with functions where the base is a constant number and the exponent is the variable. Here is an example.

$y = 2^x$

This particular function describes something that doubles each time x increases by one. Lets look at a particular situation where this might occur.
A colony of bacteria has a population of three thousand at noon on Sunday. During the next week, the colonys population doubles every day. What is the population of the bacteria colony at noon on Saturday?

Lets make a table of values and calculate the population each day.

Day	0(Sunday)	1(Monday)	2(Tuesday)	3(Wednesday)	4(Thursda
Population (in thousands)	3	6	12	24	48

To get the population of bacteria for the next day we simply multiply the current days population by 2.

$$\begin{aligned}
\text{We start with a population of 3 (thousand):} \quad & P = 3 \\
\text{To find the population on Monday we double} \quad & P = 3 \cdot 2 \\
\text{The population on Tuesday will be double that on Monday} \quad & P = 3 \cdot 2 \cdot 2 \\
\text{The population on Wednesday will be double that on Tuesday} \quad & P = 3 \cdot 2 \cdot 2 \cdot 2 \\
\text{Thursday is double that on Wednesday} \quad & P = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \\
\text{Friday is double that on Thursday} \quad & P = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \\
\text{Saturday is double that on Friday} \quad & P = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2
\end{aligned}$$

You can see that this function describes a population that is multiplied by 2 each time a day passes.

If we define x as the number of days since Sunday at noon, then we can write the following.
$P = 3.2^x$ *This is a formula that we can use to calculate the population on any day.*

For instance, the population on Saturday at noon will be $P = 3.2^6 = 3.64 = 192 \text{ (thousand)}$ bacteria.

We used $x = 6$, since Saturday at noon is six days after Sunday at noon.

In general exponential function takes the form:
$y = A \cdot b^x$ *where A is the initial amount and b is the factor that the amount gets multiplied by each time x is increased by one.*

Graph Exponential Functions

Lets start this section by graphing some exponential functions. Since we dont yet know any special properties of exponential functions, we will graph using a table of values.
Example 1
Graph the equation using a table of values $y = 2^x$.
Solution

Lets make a table of values that includes both negative and positive values of x.

To evaluate the positive values of x, we just plug into the function and evaluate.

$x = 1,$ $\quad y = 2^1 = 2$
$x = 2,$ $\quad y = 2^2 = 2 \cdot 2 = 4$
$x = 3,$ $\quad y = 2^3 = 2 \cdot 2 \cdot 2 = 8$

x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

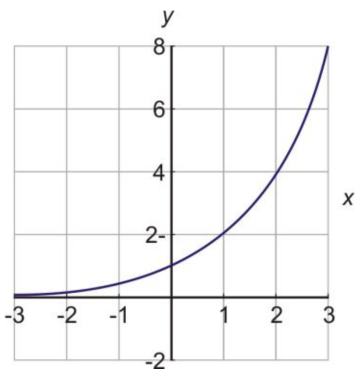
For $x = 0$, we must remember that a number to the power 0 is always 1.

$x = 0, \qquad y = 2^0 = 1$

To evaluate the negative values of x, we must remember that x to a negative power means one over x to the same positive power.

$$x = -1, \qquad y = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$$
$$x = -2, \qquad y = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$
$$x = -3, \qquad y = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

When we plot the points on the coordinate axes we get the graph below. Exponentials always have this basic shape. That is, they start very small and then, once they start growing, they grow faster and faster, and soon they become extremely big!



[Figure1]

You may have heard people say that something is growing **exponentially.** This implies that the growth is very quick. An exponential function actually starts slow, but then grows faster and faster all the time. Specifically, our function y above doubled each time we increased x by one.

This is the definition of exponential growth. There is a consistent fixed period during which the function will double or triple, or quadruple. The change is always a fixed proportion.

Compare Graphs of Exponential Growth Functions

Lets graph a few more exponential functions and see what happens as we change the constants in the functions. The basic shape of the exponential function should stay the same. But, it may become steeper or shallower depending on the constants we are using.

We mentioned that the general form of the exponential function is $y = A \cdot b^x$ where A is the initial amount and b is the factor that the amount gets multiplied by each time x is increased by one. Lets see what happens for different values of A.

Example 2

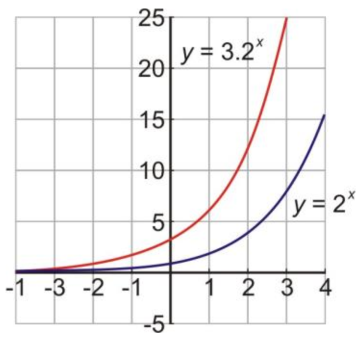
Graph the exponential function $y = 3 \cdot 2^x$ *and compare with the graph of* $y = 2^x$.

Solution

Lets make a table of values for $y = 3 \cdot 2^x$.

x	$y = 3 \cdot 2^x$
-2	$y = 3 \cdot 2^{-2} = 3 \cdot \frac{1}{2^2} = \frac{3}{4}$
-1	$y = 3 \cdot 2^{-1} = 3 \cdot \frac{1}{2^1} = \frac{3}{2}$
0	$y = 3 \cdot 2^0 = 3$
1	$y = 3 \cdot 2^1 = 6$
2	$y = 3 \cdot 2^2 = 3 \cdot 4 = 12$
3	$y = 3 \cdot 2^3 = 3 \cdot 8 = 24$

Now let's use this table to graph the function.



[Figure2]

We can see that the function $y = 3 \cdot 2^x$ is bigger than function $y = 2^x$. In both functions, the value of y doubled every time x increases by one. However, $y = 3 \cdot 2^x$ starts with a value of 3, while $y = 2^x$ starts with a value of 1, so it makes sense that $y = 3 \cdot 2^x$ would be bigger as its values of y keep getting doubled.

You might think that if the initial value A is less than one, then the corresponding exponential function would be less than $y = 2^x$. This is indeed correct. Lets see how the graphs compare for $A = \frac{1}{3}$.

Example 3

Graph the exponential function $y = \frac{1}{3} \cdot 2^x$ *and compare with the graph of* $y = 2^x$.

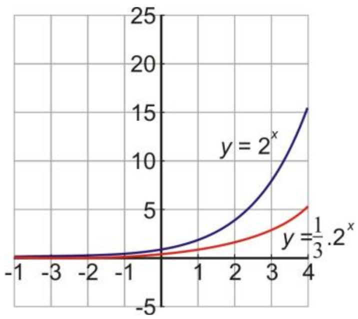
Solution

Lets make a table of values for $y = \frac{1}{3} \cdot 2^x$.

x	$y = \frac{1}{3} \cdot 2^x$

x	$y = \frac{1}{3} \cdot 2^x$
-2	$y = \frac{1}{3} \cdot 2^{-2} = \frac{1}{3} \cdot \frac{1}{2^2} = \frac{1}{12}$
-1	$y = \frac{1}{3} \cdot 2^{-1} = \frac{1}{3} \cdot \frac{1}{2^1} = \frac{1}{6}$
0	$y = \frac{1}{3} \cdot 2^0 = \frac{1}{3}$
1	$y = \frac{1}{3} \cdot 2^1 = \frac{2}{3}$
2	$y = \frac{1}{3} \cdot 2^2 = \frac{1}{3} \cdot 4 = \frac{4}{3}$
3	$y = \frac{1}{3} \cdot 2^3 = \frac{1}{3} \cdot 8 = \frac{8}{3}$

Now let's use this table to graph the function.



[Figure3]

As expected, the exponential function $y = \frac{1}{3} \cdot 2^x$ is smaller that the exponential function $y = 2^x$.

Now, lets compare exponential functions whose bases are different.

The function $y = 2^x$ has a base of 2. That means that the value of y doubles every time x is increased by 1.

The function $y = 3^x$ has a base of 3. That means that the value of y triples every time x is increased by 1.

The function $y = 5^x$ has a base of 5. That means that the value of y gets multiplies by a factor of 5 every time x is increased by 1.

The function $y = 10^x$ has a base of 10. That means that the value of y gets multiplied by a factor of 10 every time x is increased by 1.

What do you think will happen as the base number is increased? Lets find out.

Example 4

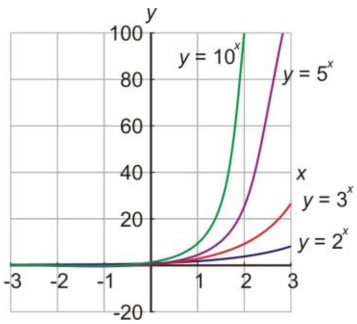
Graph the following exponential functions of the same graph $y = 2^x, y = 3^x, y = 5^x, y = 10^x$.

Solution

To graph these functions we should start by making a table of values for each of them.

x	$y = 2^x$	$y = 3^x$	$y = 5^x$	$y = 10^x$
-2	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{25}$	$\frac{1}{100}$
-1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{10}$
0	1	1	1	1
1	2	3	5	10
2	4	9	25	100
3	8	27	125	1000

Now let's graph these functions.



[Figure4]

Notice that for $x = 0$ the values for all the functions are equal to 1. This means that the initial value of the functions is the same and equal to 1. Even though all the functions start at the same value, they increase at different rates. We can see that the bigger the base is the faster the values of y will increase. It makes sense that something that triples each time will increase faster than something that just doubles each time.

Finally, lets examine what the graph of an exponential looks like if the value of A is negative.

Example 5

Graph the exponential function $y = -5 \cdot 2^x$.

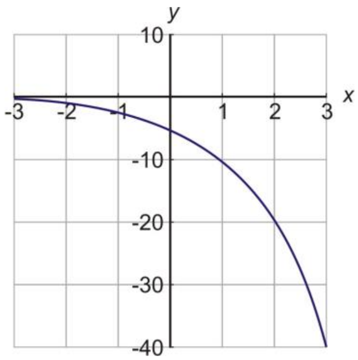
Solution

Lets make a table of values.

x $\qquad$ $y = -5 \cdot 2^x$

x	$y = -5 \cdot 2^x$
-2	$-\frac{5}{4}$
-1	$-\frac{5}{2}$
0	-5
1	-10
2	-20
3	-40

Now let's graph the function.



[Figure5]

This result should not be unexpected. Since the initial value is negative and doubles with time, it makes sense that the value of y increasess, but in a negative direction. Notice that the shape of the graph remains that of a typical exponential function, but is now a mirror image about the horizontal axis (i.e. upside down).

Solve Real-World Problems Involving Exponential Growth

We will now examine some real-world problems where exponential growth occurs.

Example 6

The population of a town is estimated to increase by 15% per year. The population today is 20 thousand. Make a graph of the population function and find out what the population will be ten years from now.

Solution

First, we need to write a function that describes the population of the town. The general form of an exponential function is.

$y = A \cdot b^x$

Define y as the population of the town.

Define x as the number of years from now.

A is the initial population, so $A = 20$ (thousand)

Finally, we must find what b is. We are told that the population increases by 15% each year.

To calculate percents, it is necessary to change them into decimals. 15% is equivalent to 0.15.

15% of A is equal to $0.15A$. This represents the increase in population from one year to the next.

In order to get the total population for the following year we must add the current population to the increase in population. In other words $A + 0.15A = 1.15A$. We see from this that the population must be multiplied by a factor of 1.15 each year.

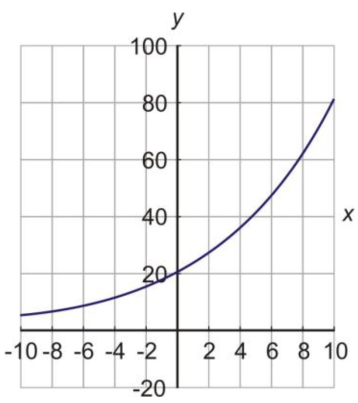
This means that the base of the exponential is $b = 1.15$.

The formula that describes this problem is $y = 20 \cdot (1.15)^x$

Now lets make a table of values.

x	$y = 20 \cdot (1.15)^x$
-10	4.9
-5	9.9
0	20
5	40.2
10	80.9

Now let's graph the function.



[Figure6]

Notice that we used negative values of x in our table of values. Does it make sense to think of negative time? In this case $x = -5$ represents what the population was five years ago, so it can be useful information. The question asked in the problem was *What will be the population of the town ten years from now?*

To find the population exactly, we use $x = 10$ in the formula. We found $y = 20 \cdot (1.15)^{10} = 89,911$.

Example 7

Peter earned $1500 last summer. If he deposited the money in a bank account that earns 5% interest compounded yearly, how much money will he have after five years?

Solution

This problem deals with interest which is compounded yearly. This means that each year the interest is calculated on the amount of money you have in the bank. That interest is added to the original amount and next year the interest is calculated on this new amount. In this way, you get paid interest on the interest.

Lets write a function that describes the amount of money in the bank. The general form of an exponential function is

$y = A \cdot b^x$

Define y as the amount of money in the bank.

Define x as the number of years from now.

A is the initial amount, so $A = 1500$.

Now we must find what b is.

We are told that the interest is 5% each year.

Change percents into decimals 5% is equivalent to 0.05.
5% of A is equal to $0.05A$ *This represents the interest earned per year.*

In order to get the total amount of money for the following year, we must add the interest earned to the initial amount.

$A + 0.05A = 1.05A$

We see from this that the amount of money must be multiplied by a factor of 1.05 each year.

This means that the base of the exponential is $b = 1.05$

The formula that describes this problem is $y = 1500 \cdot (1.05)^x$

To find the total amount of money in the bank at the end of five years, we simply use $x = 5$ in our formula.

Answer $y = 1500 \cdot (1.05)^5 = \$1914.42$

Review Questions

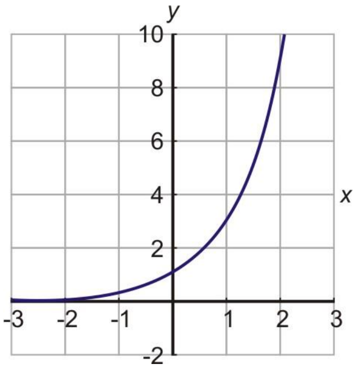
Graph the following exponential functions by making a table of values.

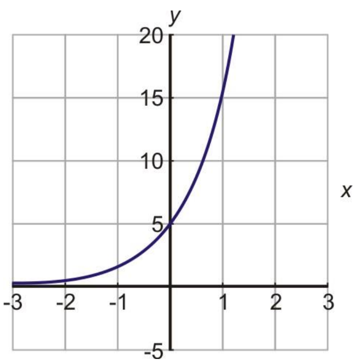
1. $y = 3^x$
2. $y = 5 \cdot 3^x$
3. $y = 40 \cdot 4^x$
4. $y = 3 \cdot 10^x$

Solve the following problems.

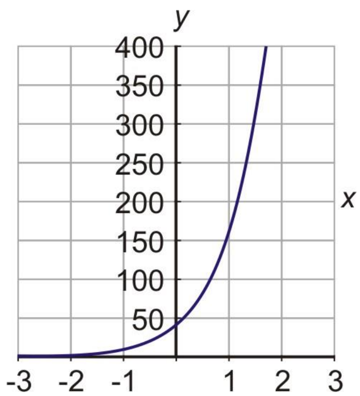
5. A chain letter is sent out to 10 people telling everyone to make 10 copies of the letter and send each one to a new person. Assume that everyone who receives the letter sends it to ten new people and that it takes a week for each cycle. How many people receive the letter on the sixth week?
6. Nadia received $200 for her 10^{th} birthday. If she saves it in a bank with a 7.5% interest compounded yearly, how much money will she have in the bank by her 21^{st} birthday?

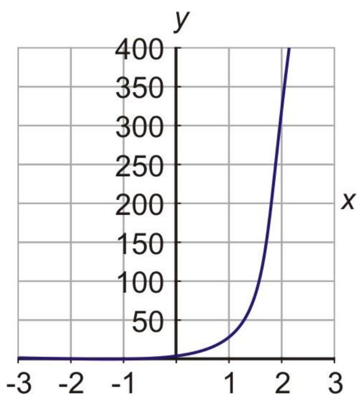
Review Answers

1. 

2. 

[Figure7][Figure8][Figure9][Figure10]

3. 

4. 

5. 10,000,000
6. $443.12

Image Attributions