Linear, Exponential, and Quadratic Models

What if you were given a table of *x* and *y* values? How could you determine if those values represented a linear function, an exponential function, or a quadratic function? After completing this Concept, you'll be able to identify functions using differences and ratios between their values.

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CK-12 Foundation: 1011S Linear, Exponential and Quadratic Models

Guidance

In this course we've learned about three types of functions, linear, quadratic and exponential.

- Linear functions take the form $y = mx + b$.
- Quadratic functions take the form $y = ax^2 + bx + c$.
- Exponential functions take the form $y = a \cdot b^x$.

In real-world applications, the function that describes some physical situation is not given; it has to be found before the problem can be solved. For example, scientific data such as observations of planetary motion are often collected as a set of measurements given in a table. Part of the scientist's job is to figure out which function best fits the data. In this section, you'll learn some methods that are used to identify which function describes the relationship between the variables in a problem.

Identify Functions Using Differences or Ratios

One method for identifying functions is to look at the difference or the ratio of different values of the dependent variable. For example, **if the difference between values of the dependent variable is the same each time we change the independent variable by the same amount, then the function is *linear* .**

Example A

Determine if the function represented by the following table of values is linear.

x	y
-2	–4

x	y
–1	–1
0	2
1	5
2	8

If we take the difference between consecutive $y-$ values, we see that each time the $x-$ value increases by one, the $y-$ value always increases by 3.

Since the difference is always the same, **the function is linear.**

When we look at the difference of the $y-$ values, we have to make sure that we examine entries for which the $x-$ values increase by the same amount.

For example, examine the values in this table:

x	y
0	5
1	10
3	20
4	25
6	35

At first glance this function might not look linear, because the difference in the $y-$ values is not always the same. But if we look closer, we can see that when the $y-$ value increases by 10 instead of 5, it's because the $x-$ value increased by 2 instead of 1. Whenever the $x-$ value increases by the *same* amount, the $y-$ value does too, so the function is linear.

Another way to think of this is in mathematical notation. We can say that a function is linear if $\frac{y_2 - y_1}{x_2 - x_1}$ is always the same for any two pairs of $x-$ and $y-$ values. Notice that the expression we used here is simply the definition of the slope of a line.

Differences can also be used to identify quadratic functions. **For a quadratic function, when we increase the $x-$ values by the same amount, the difference between $y-$ values will not be the same. However, the difference of the *differences* of the $y-$ values will be the same.**

Here are some examples of quadratic relationships represented by tables of values:

x	$y = x^2$	difference of $y-$ values	difference of differences
0	0	1 – 0 = **1**	
1	1	4 – 1 = **3**	3 – 1 = **2**
2	4	9 – 4 = **5**	5 – 3 = **2**
3	9	16 – 9 = **7**	7 – 5 = **2**
4	16	25 – 16 = **9**	9 – 7 = **2**
5	25	36 – 25 = **11**	11 – 9 = **2**
6	36		

In this quadratic function, $y = x^2$, when we increase the $x-$ value by one, the value of $y$increases by different values. However, it increases at a constant rate, so the difference of the difference is always 2.

x	$y = 2x^2$ $3x + 1$	difference of $y-$ values	difference of differences
0	0	0 – 1 = **-1**	
1	1	3 – 0 = **3**	3 + 1 = **4**
2	3	10 – 3 = **7**	7 – 3 = **4**
3	10	21 – 10 = **11**	11 – 7 = **4**
4	21	36 – 21 = **15**	15 – 11 = **4**
5	36	55 – 36 = **19**	19 – 15 = **4**
6	55		

In this quadratic function, $y = 2x^2 - 3x + 1$, when we increase the $x-$ value by one, the value of y increases by different values. However, the increase is constant: the difference of the difference is always 4.

To identify exponential functions, we use ratios instead of differences. **If the ratio between values of the dependent variable is the same each time we change the independent variable by the same amount, then the function is *exponential.***

Example B

Determine if the function represented by each table of values is exponential.

a)

x	y	ratio of y - values
0	4	$\frac{12}{4} = 3$
1	12	$\frac{36}{12} = 3$
2	36	$\frac{108}{36} = 3$
3	108	$\frac{324}{108} = 3$
4	324	

b)

x	y	ratio of y - values
0	240	$\frac{120}{240} = \frac{1}{2}$
1	120	$\frac{60}{120} = \frac{1}{2}$
2	60	$\frac{30}{60} = \frac{1}{2}$
3	30	$\frac{15}{30} = \frac{1}{2}$
4	15	

a) If we take the ratio of consecutive $y-$ values, we see that each time the $x-$ value increases by one, the $y-$ value is multiplied by 3. Since the ratio is always the same, **the function is exponential.**

b) If we take the ratio of consecutive $y-$ values, we see that each time the $x-$ value increases by one, the $y-$ value is multiplied by $\frac{1}{2}$. Since the ratio is always the same, **the function is exponential.**

Write Equations for Functions

Once we identify which type of function fits the given values, we can write an equation for the function by starting with the general form for that type of function.

Example C

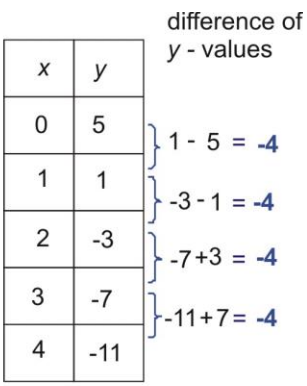
Determine what type of function represents the values in the following table.

x	y
0	5
1	1

x	y
2	-3
3	-7
4	-11

Solution

Let’s first check the difference of consecutive values of y .



x	y
0	5
1	1
2	-3
3	-7
4	-11

If we take the difference between consecutive $y-$ values, we see that each time the $x-$ value increases by one, the $y-$ value always decreases by 4. Since the difference is always the same,**the function is linear.**

To find the equation for the function, we start with the general form of a linear function: $y = mx + b$. Since m is the slope of the line, it’s also the quantity by which y increases every time the value of x increases by one. The constant b is the value of the function when $x = 0$. Therefore, the function is $y = -4x + 5$.

Example D

Determine what type of function represents the values in the following table.

x	y
0	0
1	5

x	y
2	20
3	45
4	80
5	125
6	180

Solution

Here, the difference between consecutive $y-$ values isn't constant, so the function is not linear. Let's look at those differences more closely.

x	y	
0	0	
1	5	$5-0=5$
2	20	$20-5=15$
3	45	$45-20=25$
4	80	$80-45=35$
5	125	$125-80=45$
6	180	$180-125=55$

When the $x-$ value increases by one, the difference between the $y-$ values increases by 10 every time. Since the difference of the differences is constant, the function describing this set of values is **quadratic.**

To find the equation for the function that represents these values, we start with the general form of a quadratic function: $y = ax^2 + bx + c$.

We need to use the values in the table to find the values of the constants: a, b and c .

The value of c represents the value of the function when $x = 0$, so $c = 0$.

$$\begin{aligned} \text{Plug in the point } (1,5): &\quad 5 = a + b \\ \text{Plug in the point } (2,20): &\quad 20 = 4a + 2b \Rightarrow 10 = 2a + b \\ \text{To find } a \text{ and } b, \text{we solve the system of equations:} &\quad 5 = a + b \\ &\quad 10 = 2a + b \\ \text{Solve the first equation for } b: &\quad 5 = a + b \Rightarrow b = 5 - a \\ \text{Plug the first equation into the second:} &\quad 10 = 2a + 5 - a \\ \text{Solve for } a \text{ and } b &\quad a = 5 \text{ and } b = 0 \end{aligned}$$

Therefore the equation of the quadratic function is $y = 5x^2$.

Watch this video for help with the Examples above.

CK-12 Foundation: 1011 Linear, Exponential and Quadratic Models

Vocabulary

- If the differences of the $y-$ values is always the same, **the function is linear.**
- If the difference of the *differences* of the $y-$ values is always the same, **the function is quadratic.**
- If the ratio of the $y-$ values is always the same, **the function is exponential.**

Guided Practice

Determine what type of function represents the values in the following table.

x	y
0	400
1	500
2	25

x	y
3	6.25
4	1.5625

Solution

The differences between consecutive $y-$ values aren’t the same, and the differences between those differences aren’t the same either. So let’s check the ratios instead.

x	y	ratio of y - values
0	400	
1	100	$\frac{100}{400} = \frac{1}{4}$
2	25	$\frac{25}{100} = \frac{1}{4}$
3	6.25	$\frac{6.25}{25} = \frac{1}{4}$
4	1.5625	$\frac{1.5625}{6.25} = \frac{1}{4}$

Each time the $x-$ value increases by one, the $y-$ value is multiplied by $\frac{1}{4}$. Since the ratio is always the same, **the function is exponential.**

To find the equation for the function that represents these values, we start with the general form of an exponential function, $y = a \cdot b^x$.

Here b is the ratio between the values of y each time x is increased by one. The constant a is the value of the function when $x = 0$. Therefore, the function is $y = 400\left(\frac{1}{4}\right)^x$.

Practice

Determine whether the data in the following tables can be represented by a linear function.

x	y
-4	10

x	y
-3	7
-2	4
-1	1
0	-2
1	-5

x	y
-2	4
-1	3
0	2
1	3
2	6
3	11

x	y
0	50
1	75
2	100

x	y
3	125
4	150
5	175

Determine whether the data in the following tables can be represented by a quadratic function.

x	y
-10	10
-5	2.5
0	0
5	2.5
10	10
15	22.5

x	y
1	4
2	6
3	6
4	4

x	y
5	0
6	-6

x	y
-3	-27
-2	-8
-1	-1
0	0
1	1
2	8
3	27

Determine whether the data in the following tables can be represented by an exponential function.

x	y
0	200
1	300
2	1800
3	8300

x	y
4	25800
5	62700

x	y
0	120
1	180
2	270
3	405
4	607.5
5	911.25

x	y
0	4000
1	2400
2	1440
3	864
4	518.4
5	311.04

Determine what type of function represents the values in the following tables and find the equation of each function.

x	y
0	400
1	500
2	625
3	781.25
4	976.5625

x	y
-9	-3
-7	-2
-5	-1
-3	0
-1	1
1	2

x	y
-3	14
-2	4

x	y
-1	-2
0	-4
1	-2
2	4
3	14