Solving Exponential Equations

"I'm thinking of a number," you tell your best friend. "The number I'm thinking of satisfies the equation $4^{x+1} = 256$. What number are you thinking of?

Guidance

Until now, we have only solved pretty basic exponential equations, like #1 in the Review Queue above. We know that $x = 5$, because $2^5 = 32$. Ones like #4 are a little more challenging, but if we put everything into a power of 2, we can set the exponents equal to each other and solve.

$$\begin{aligned} 8^x &= 128 \\ 2^{3x} &= 2^7 \\ 3x &= 7 \\ x &= \frac{7}{3} \end{aligned}$$

So, $8^{\frac{7}{3}} = 128$.

But, what happens when the power is not easily found? We must use logarithms, followed by the Power Property to solve for the exponent.

Example A

Solve $6^x = 49$. Round your answer to the nearest three decimal places.

Solution: To solve this exponential equation, let's take the logarithm of both sides. The easiest logs to use are either $\ln$ (the natural log), or log (log, base 10). We will use the natural log.

$$\begin{aligned} 6^x &= 49 \\ \ln 6^x &= \ln 49 \\ x \ln 6 &= \ln 49 \\ x &= \frac{\ln 49}{\ln 6} \approx 2.172 \end{aligned}$$

Example B

Solve $10^{x-3} = 100^{3x+11}$.

Solution: Change 100 into a power of 10.

$$10^{x-3} = 10^{2(3x+11)}$$
$$x - 3 = 6x + 22$$
$$-25 = 5x$$
$$-5 = x$$

Example C

Solve $8^{2x-3} - 4 = 5$.

Solution: Add 4 to both sides and then take the log of both sides.

$$8^{2x-3} - 4 = 5$$
$$8^{2x-3} = 9$$
$$\log 8^{2x-3} = \log 9$$
$$(2x-3)\log 8 = \log 9$$
$$2x - 3 = \frac{\log 9}{\log 8}$$
$$2x = 3 + \frac{\log 9}{\log 8}$$
$$x = \frac{3}{2} + \frac{\log 9}{2\log 8} \approx 2.56$$

Notice that we did not find the numeric value of $\log 9$ or $\log 8$ until the very end. This will ensure that we have the most accurate answer.

Intro Problem Revisit We can rewrite the equation $4^{x+1} = 256$ as $2^{2(x+1)} = 2^8$ and solve for *x* .

$$2^{2(x+1)} = 2^8$$
$$2^{2x+2} = 2^8$$
$$2x + 2 = 8$$
$$x = 3$$

Therefore, you're thinking of the number 3.

Guided Practice

Solve the following exponential equations.

1. $4^{x-8} = 16$
2. $2(7)^{3x+1} = 48$

3. $\frac{2}{3} \cdot 5^{x+2} + 9 = 21$

Answers

1. Change 16 to 4^2 and set the exponents equal to each other.

$$\begin{aligned} 4^{x-8} &= 16 \\ 4^{x-8} &= 4^2 \\ x - 8 &= 2 \\ x &= 10 \end{aligned}$$

2. Divide both sides by 2 and then take the log of both sides.

$$\begin{aligned} 2(7)^{3x+1} &= 48 \\ 7^{3x+1} &= 24 \\ \ln 7^{3x+1} &= \ln 24 \\ (3x+1)\ln 7 &= \ln 24 \\ 3x + 1 &= \frac{\ln 24}{\ln 7} \\ 3x &= -1 + \frac{\ln 24}{\ln 7} \\ x &= -\frac{1}{3} + \frac{\ln 24}{3 \ln 7} \approx 0.211 \end{aligned}$$

3. Subtract 9 from both sides and multiply both sides by $\frac{3}{2}$. Then, take the log of both sides.

$$\begin{aligned} \frac{2}{3} \cdot 5^{x+2} + 9 &= 21 \\ \frac{2}{3} \cdot 5^{x+2} &= 12 \\ 5^{x+2} &= 18 \\ (x+2)\log 5 &= \log 18 \\ x &= \frac{\log 18}{\log 5} - 2 \approx -0.204 \end{aligned}$$

Practice

Use logarithms and a calculator to solve the following equations for x . Round answers to three decimal places.

1. $5^x = 65$
2. $7^x = 75$
3. $2^x = 90$
4. $3^{x-2} = 43$
5. $6^{x+1} + 3 = 13$
6. $6(11^{3x-2}) = 216$
7. $8 + 13^{2x-5} = 35$
8. $\frac{1}{2} \cdot 7^{x-3} - 5 = 14$

Solve the following exponential equations without a calculator.

9. $4^x = 8$
10. $9^{x-2} = 27$
11. $5^{2x+1} = 125$
12. $9^3 = 3^{4x-6}$
13. $7(2^{x-3}) = 56$
14. $16^x \cdot 4^{x+1} = 32^{x+1}$
15. $3^{3x+5} = 3 \cdot 9^{x+3}$