Trigonometric Ratios on the Unit Circle

What are the exact values of the following trigonometric functions?

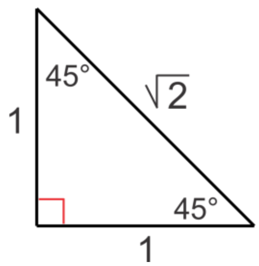
a. $\sin 495^\circ$

b. $\tan \frac{5\pi}{3}$

Guidance

Recall special right triangles from Geometry. In a $(30^\circ - 60^\circ - 90^\circ)$ triangle, the sides are in the ratio $1 : \sqrt{3} : 2$.

In an isosceles triangle $(45^\circ - 45^\circ - 90^\circ)$, the congruent sides and the hypotenuse are in the ratio $1 : 1 : \sqrt{2}$.



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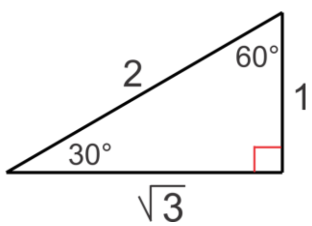
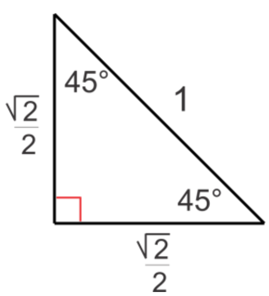
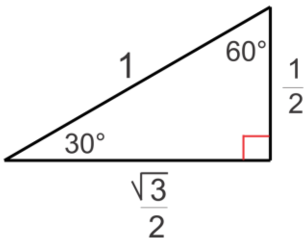


Now let's make the hypotenuse equal to 1 in each of the triangles so we'll be able to put them inside the unit circle. Using the appropriate ratios, the new side lengths are:





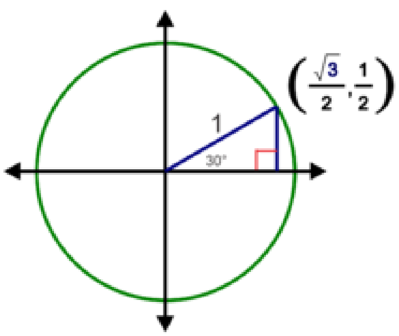
Using these triangles, we can evaluate sine, cosine and tangent for each of the angle measures.

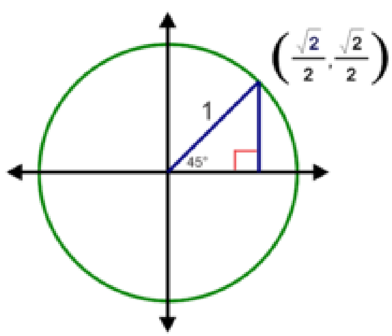
$$\sin 45^\circ = \frac{\sqrt{2}}{2} \qquad \sin 60^\circ = \frac{\sqrt{3}}{2} \qquad \sin 30^\circ = \frac{1}{2}$$

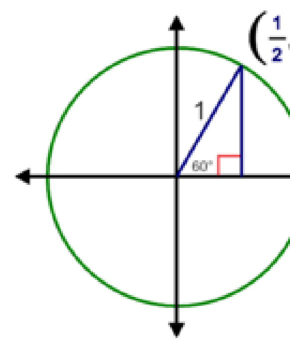
$$\cos 45^\circ = \frac{\sqrt{2}}{2} \qquad \cos 60^\circ = \frac{1}{2} \qquad \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\tan 45^\circ = 1 \qquad \tan 60^\circ = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \qquad \tan 30^\circ = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{2}$$

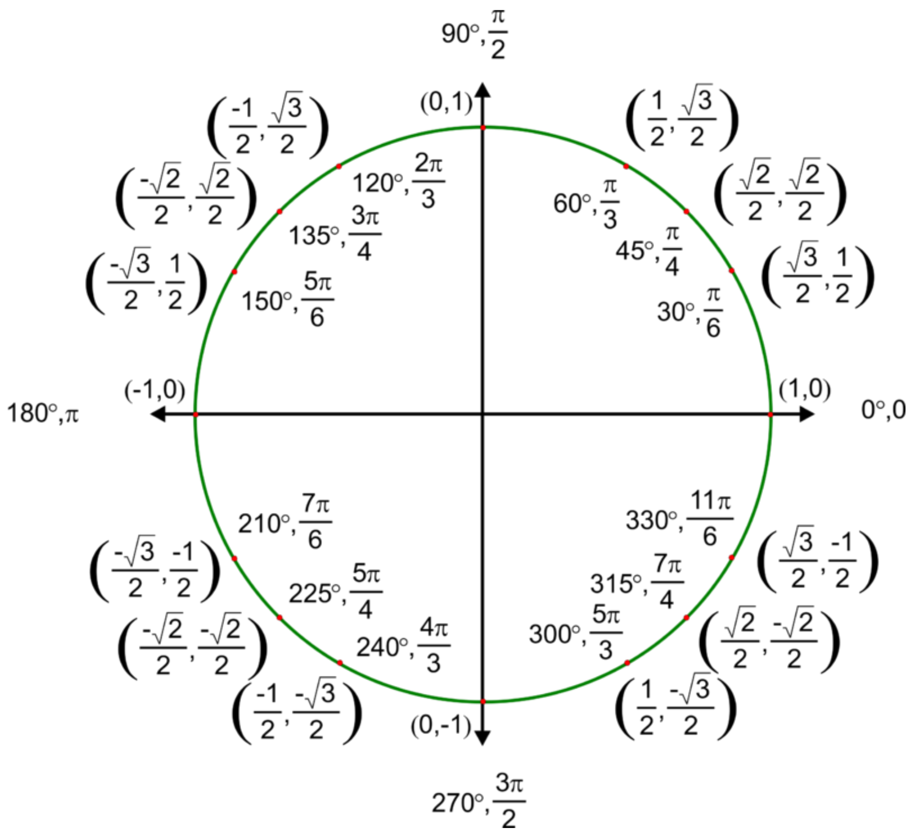
These triangles can now fit inside the unit circle.







Putting together the trigonometric ratios and the coordinates of the points on the circle, which represent the lengths of the legs of the triangles, $(\Delta x, \Delta y)$, we can see that each point is actually $(\cos\theta, \sin\theta)$, where θ is the reference angle. For example, $\sin 60^\circ = \frac{\sqrt{3}}{2}$ is the y – coordinate of the point on the unit circle in the triangle with reference angle 60° . By reflecting these triangles across the axes and finding the points on the axes, we can find the trigonometric ratios of all multiples of $0^\circ, 30^\circ$ and 45° (or $0, \frac{\pi}{6}, \frac{\pi}{4}$ radians).



Example A

Find $\sin \frac{3\pi}{2}$.

Solution: Find $\frac{3\pi}{2}$ on the unit circle and the corresponding point is $(0, -1)$. Since each point on the unit circle is $(\cos \theta, \sin \theta), \sin \frac{3\pi}{2} = -1$.

Example B

Find $\tan \frac{7\pi}{6}$.

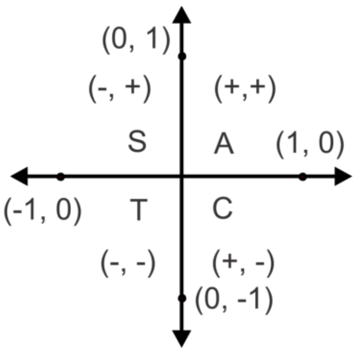
Solution: This time we need to look at the ratio $\frac{\sin \theta}{\cos \theta}$. We can use the unit circle to find $\sin \frac{7\pi}{6} = -\frac{1}{2}$ and $\cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$. Now, $\tan \frac{7\pi}{6} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

More Guidance

Another way to approach these exact value problems is to use the reference angles and the special right triangles. The benefit of this method is that there is no need to memorize the entire unit circle. If you memorize the special right triangles, can determine reference angles and know where the ratios are positive and negative you can put the pieces together to get the ratios. Looking at the unit circle above, we see that all of the ratios are positive in Quadrant I, sine is the only positive ratio in Quadrant II, tangent is the only positive ratio in Quadrant III and cosine is the only positive ratio in Quadrant IV.

Keeping this diagram in mind will help you remember where cosine, sine and tangent are positive and negative. You can also use the pneumonic device - **A** ll **S** tudents **T** ake **C** alculus, or **ASTC** , to recall which is positive (all the others would be negative) in which quadrant.

The coordinates on the vertices will help you determine the ratios for the multiples of 90° or $\frac{\pi}{2}$.



Example C

Find the exact values for the following trigonometric functions using the alternative method.

a. $\cos 120^\circ$

b. $\sin \frac{5\pi}{3}$

c. $\tan \frac{7\pi}{2}$

Solution:

a. First, we need to determine in which quadrant the angles lies. Since 120° is between 90° and 180° it will lie in Quadrant II. Next, find the reference angle. Since we are in QII, we will subtract from 180° to get 60° . We can use the reference angle to find the ratio, $\cos 60^\circ = \frac{1}{2}$. Since we are in QII where only sine is positive, $\cos 120^\circ = -\frac{1}{2}$.

b. This time we will need to work in terms of radians but the process is the same. The angle $\frac{5\pi}{3}$ lies in QIV and the reference angle is $\frac{\pi}{3}$. This means that our ratio will be negative. Since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, \sin \frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$.

c. The angle $\frac{7\pi}{2}$ represents more than one entire revolution and it is equivalent to $2\pi + \frac{3\pi}{2}$. Since our angle is a multiple of $\frac{\pi}{2}$ we are looking at an angle on an axis. In this case, the point is $(0, -1)$. Because $\tan\theta = \frac{\sin\theta}{\cos\theta}, \tan\frac{7\pi}{2} = \frac{-1}{0}$, which is undefined. Thus, $\tan\frac{7\pi}{2}$ is undefined.

Concept Problem Revisit

a. First, we need to determine in which quadrant the angle lies.
Since $495^\circ - 360^\circ = 135^\circ$ is between 90° and 180° it will lie in Quadrant II. Next, find the reference angle. Since we are in QII, we will subtract from 180° to get 45°. We can use the reference angle to find the ratio, $\cos 45^\circ = \frac{\sqrt{2}}{2}$. Since we are in QII where only sine is positive, $\cos 495^\circ = -\frac{\sqrt{2}}{2}$.

b. In the previous example we established that the angle $\frac{5\pi}{3}$ lies in QIV and the reference angle is $\frac{\pi}{3}$. This means that the tangent ratio will be negative.
Since $\tan\frac{\pi}{3} = \sqrt{3}, \tan\frac{5\pi}{3} = -\sqrt{3}$.

Guided Practice

Find the exact trigonometric ratios. You may use either method.

1. $\cos\frac{7\pi}{3}$
2. $\tan\frac{9\pi}{2}$
3. $\sin 405^\circ$
4. $\tan\frac{11\pi}{6}$
5. $\cos\frac{2\pi}{3}$

Answers

1. $\frac{7\pi}{3}$ has a reference angle of $\frac{\pi}{3}$ in QI. $\cos\frac{\pi}{3} = \frac{1}{2}$ and since cosine is positive in QI, $\cos\frac{7\pi}{3} = \frac{1}{2}$.

2. $\frac{9\pi}{2}$ is coterminal to $\frac{\pi}{2}$ which has coordinates (0, 1). So $\tan\frac{9\pi}{2} = \frac{\sin\frac{9\pi}{2}}{\cos\frac{9\pi}{2}} = \frac{1}{0}$ which is undefined.

3. 405° has a reference angle of 45° in QI. $\sin 45^\circ = \frac{\sqrt{2}}{2}$ and since sine is positive in QI, $\sin 405^\circ = \frac{\sqrt{2}}{2}$.

4. $\frac{11\pi}{6}$ is coterminal to $\frac{\pi}{6}$ in QIV. $\tan\frac{\pi}{6}=\frac{\sqrt{3}}{3}$ and since tangent is negative in QIV, $\tan\frac{11\pi}{6}=-\frac{\sqrt{3}}{3}$.

5. $\frac{2\pi}{3}$ is coterminal to $\frac{\pi}{3}$ in QII. $\cos\frac{\pi}{3}=\frac{1}{2}$ and since cosine is negative in QII, $\cos\frac{2\pi}{3}=\frac{1}{2}$.

Practice

Find the exact values for the following trigonometric functions.

1. $\sin\frac{3\pi}{4}$
2. $\cos\frac{3\pi}{2}$
3. $\tan 300^\circ$
4. $\sin 150^\circ$
5. $\cos\frac{4\pi}{3}$
6. $\tan\pi$
7. $\cos\left(-\frac{15\pi}{4}\right)$
8. $\sin 225^\circ$
9. $\tan\frac{7\pi}{6}$
10. $\sin 315^\circ$
11. $\cos 450^\circ$
12. $\sin\left(-\frac{7\pi}{2}\right)$
13. $\cos\frac{17\pi}{6}$
14. $\tan 270^\circ$
15. $\sin(-210^\circ)$