Pythagorean Identities

What if you were working on a problem using the unit circle and had the value of one trig function (such as sine), but wanted instead to find the value of another trig function (such as cosine)? Is this possible?

Try it with $\sin \theta = \frac{1}{2}$

Keep reading, and when this Concept is finished, you'll know how to use this information to help you find $\cos \theta$.

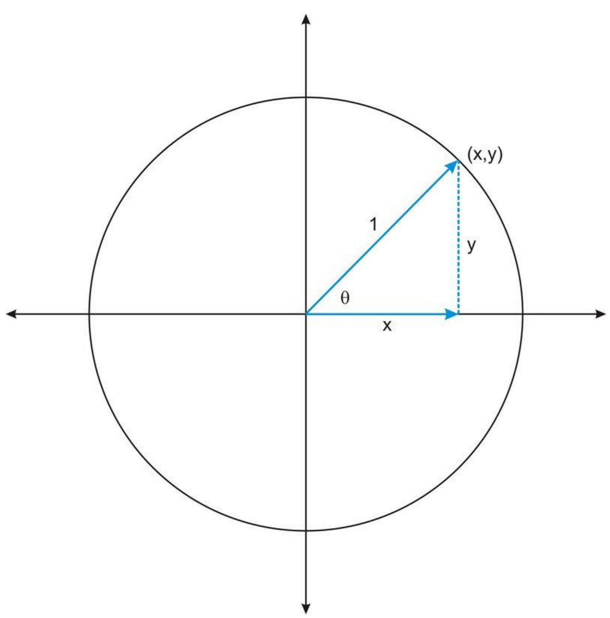
Watch This

The final portion of this video reviews the Pythagorean Identities.

James Sousa: The Reciprocal, Quotient, and Pythagorean Identities

Guidance

One set of identities are called the Pythagorean Identities because they rely on the Pythagorean Theorem. In other Concepts we used the Pythagorean Theorem to find the sides of right triangles. Consider the way that the trig functions are defined. Let's look at the unit circle:



The legs of the right triangle are x and y . The hypotenuse is 1. Therefore the following equation is true for all x and y on the unit circle:

$x^2 + y^2 = 1$

Now remember that on the unit circle, $\cos\theta = x$ and $\sin\theta = y$. Therefore the following equation is an identity:

$\cos^2\theta + \sin^2\theta = 1$

Note: Writing the exponent 2 after the cos and sin is the standard way of writing exponents. Just keeping mind that $\cos^2\theta$ means $(\cos\theta)^2$ and $\sin^2\theta$ means $(\sin\theta)^2$.

We can use this identity to find the value of the sine function, given the value of the cosine, and vice versa. We can also use it to find other identities.

Example A

If $\cos\theta = \frac{1}{4}$ what is the value of $\sin\theta$? Assume that θ is an angle in the first quadrant.

Solution: $\sin\theta = \frac{\sqrt{15}}{4}$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\left(\frac{1}{4}\right)^2 + \sin^2\theta = 1$$

$$\frac{1}{16} + \sin^2\theta = 1$$

$$\sin^2\theta = 1 - \frac{1}{16}$$

$$\sin^2\theta = \frac{15}{16}$$

$$\sin\theta = \pm\sqrt{\frac{15}{16}}$$

$$\sin\theta = \pm\frac{\sqrt{15}}{4}$$

Remember that it was given that θ is an angle in the first quadrant. Therefore the sine value is positive, so $\sin\theta = \frac{\sqrt{15}}{4}$.

Example B

Use the identity $\cos^2\theta + \sin^2\theta = 1$ to show that $\cot^2\theta + 1 = \csc^2\theta$

Solution:

$$\cos^2\theta + \sin^2\theta = 1 \qquad \text{Divide both sides by} \sin^2\theta.$$

$$\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \qquad \frac{\sin^2\theta}{\sin^2\theta} = 1$$

$$\frac{\cos^2\theta}{\sin^2\theta} + 1 = \frac{1}{\sin^2\theta}$$

$$\frac{\cos\theta}{\sin\theta} \times \frac{\cos\theta}{\sin\theta} + 1 = \frac{1}{\sin\theta} \times \frac{1}{\sin\theta} \qquad \text{Write the squared functions in terms of their factors.}$$

$$\cot\theta \times \cot\theta + 1 = \csc\theta \times \csc\theta \qquad \text{Use the quotient and reciprocal identities.}$$

$$\cot^2\theta + 1 = \csc^2\theta \qquad \text{Write the functions as squared functions.}$$

Example C

If $\sin\theta = \frac{1}{2}$ what is the value of $\cos\theta$? Assume that θ is an angle in the first quadrant.

Solution: $\cos\theta = \sqrt{\frac{3}{4}}$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\left(\frac{1}{2}\right)^2 + \cos^2\theta = 1$$

$$\frac{1}{4} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{1}{4}$$

$$\cos^2\theta = \frac{3}{4}$$

$$\cos\theta = \pm\sqrt{\frac{3}{4}}$$

Remember that it was given that θ is an angle in the first quadrant. Therefore the cosine value is positive, so $\cos\theta = \sqrt{\frac{3}{4}}$.

Vocabulary

Pythagorean Identity: A ***pythagorean identity*** is a relationship showing that the sine of an angle squared plus the cosine of an angle squared is equal to one.

Guided Practice

1. If $\cos\theta = \frac{1}{2}$ what is the value of $\sin\theta$? Assume that θ is an angle in the first quadrant.

2. If $\sin\theta = \frac{1}{8}$ what is the value of $\cos\theta$? Assume that θ is an angle in the first quadrant.

3. If $\sin\theta = \frac{1}{3}$ what is the value of $\cos\theta$? Assume that θ is an angle in the first quadrant.

Solutions:

1. The solution is $\sin\theta = \sqrt{\frac{3}{4}}$. We can see this from the Pythagorean Identity:

$$\cos^2\theta + \sin^2\theta = 1$$

$$\left(\frac{1}{2}\right)^2 + \sin^2\theta = 1$$

$$\frac{1}{4} + \sin^2\theta = 1$$

$$\sin^2\theta = 1 - \frac{1}{4}$$

$$\sin^2\theta = \frac{3}{4}$$

$$\sin\theta = \pm\sqrt{\frac{3}{4}}$$

2. The solution is $\cos\theta = \sqrt{\frac{63}{64}}$. We can see this from the Pythagorean Identity:

$$\cos^2\theta + \sin^2\theta = 1$$

$$\left(\frac{1}{8}\right)^2 + \cos^2\theta = 1$$

$$\frac{1}{64} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{1}{64}$$

$$\cos^2\theta = \frac{63}{64}$$

$$\cos\theta = \pm\sqrt{\frac{63}{64}}$$

3. The solution is $\cos\theta = \sqrt{\frac{8}{9}}$. We can see this from the Pythagorean Identity:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\left(\frac{1}{3}\right)^2 + \cos^2\theta = 1$$

$$\frac{1}{9} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{1}{9}$$

$$\cos^2\theta = \frac{8}{9}$$

$$\cos\theta = \pm\sqrt{\frac{8}{9}}$$

Concept Problem Solution

Since we now know that:

$$\sin^2\theta + \cos^2\theta = 1$$

we can use this to help us compute the cosine of the angle from the problem at the beginning of this Concept. It was given at the beginning of this Concept that:

$$\sin\theta = \frac{1}{2}$$

Therefore, $\sin^2\theta = \frac{1}{4}$

If we use this to solve for cosine:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \sin^2\theta$$
$$\cos^2\theta = 1 - \frac{1}{4}$$
$$\cos^2\theta = \frac{3}{4}$$
$$\cos\theta = \frac{\sqrt{3}}{2}$$

Practice

1. If you know $\sin\theta$, what other trigonometric value can you determine using a Pythagorean Identity?
2. If you know $\sec\theta$, what other trigonometric value can you determine using a Pythagorean Identity?
3. If you know $\cot\theta$, what other trigonometric value can you determine using a Pythagorean Identity?
4. If you know $\tan\theta$, what other trigonometric value can you determine using a Pythagorean Identity?

For questions 5-14, assume all angles are in the first quadrant.

5. If $\sin\theta = \frac{1}{2}$, what is the value of $\cos\theta$?
6. If $\cos\theta = \frac{\sqrt{2}}{2}$, what is the value of $\sin\theta$?
7. If $\tan\theta = 1$, what is the value of $\sec\theta$?
8. If $\csc\theta = \sqrt{2}$, what is the value of $\cot\theta$?
9. If $\sec\theta = 2$, what is the value of $\tan\theta$?
10. If $\cot\theta = \sqrt{3}$, what is the value of $\csc\theta$?
11. If $\cos\theta = \frac{1}{4}$, what is the value of $\sin\theta$?
12. If $\sec\theta = 3$, what is the value of $\tan\theta$?
13. If $\sin\theta = \frac{1}{5}$, what is the value of $\cos\theta$?
14. If $\tan\theta = \frac{\sqrt{3}}{3}$, what is the value of $\sec\theta$?
15. Use the identity $\sin^2\theta + \cos^2\theta = 1$ to show that $\tan^2\theta + 1 = \sec^2\theta$