Sine Sum and Difference Formulas

You've gotten quite good at knowing the values of trig functions. So much so that you and your friends play a game before class everyday to see who can get the most trig functions of differentangles correct. However, your friend Jane keeps getting the trig functions of more angles right. You're amazed by her memory, until she smiles one day and tells you that she's been fooling you all this time.

"What you do you mean?" you say.

"I have a trick that lets me calculate more functions in my mind by breaking them down into sums of angles." she replies.

You're really surprised by this. And all this time you thought she just had an amazing memory!

"Here, let me show you," she says. She takes a piece of paper out and writes down:

$\sin \frac{7\pi}{12}$

"This looks like an unusual value to remember for a trig function. So I have a special rule that helps me to evaluate it by breaking it into a sum of different numbers."

By the end of this Concept, you'll be able to calculate the above function using a special rule, just like Jane does.

Watch This

James Sousa: Sum and Difference Identities for Sine

Guidance

Our goal here is to figure out a formula that lets you break down a the sine of a sum of two angles(or a difference of two angles) into a simpler formula that lets you use the sine of only one argument in each term.

To find $\sin(a+b)$:

$$\begin{aligned}
\sin(a+b) &= \cos\left[\frac{\pi}{2}-(a+b)\right] && \text{Set } \theta = a+b \\
&= \cos\left[\left(\frac{\pi}{2}-a\right)-b\right] && \text{Distribute the negative} \\
&= \cos\left(\frac{\pi}{2}-a\right)\cos b + \sin\left(\frac{\pi}{2}-a\right)\sin b && \text{Difference Formula for cosines} \\
&= \sin a \cos b + \cos a \sin b && \text{Co-function Identities}
\end{aligned}$$

In conclusion, $\sin(a+b) = \sin a \cos b + \cos a \sin b$, which is the *sum* formula for sine.

To obtain the identity for $\sin(a-b)$:

$$\begin{aligned}
\sin(a-b) &= \sin[a+(-b)] \\
&= \sin a \cos(-b) + \cos a \sin(-b) \quad \text{Use the sine sum formula} \\
\sin(a-b) &= \sin a \cos b - \cos a \sin b \quad \text{Use } \cos(-b) = \cos b, \text{ and } \sin(-b) = -\sin b
\end{aligned}$$

In conclusion, $\sin(a-b) = \sin a \cos b - \cos a \sin b$, so, this is the *difference* formula for sine.

Example A

Find the exact value of $\sin \frac{5\pi}{12}$

Solution: Recall that there are multiple angles that add or subtract to equal any angle. Choose whichever formula that you feel more comfortable with.

$$\begin{aligned}
\sin \frac{5\pi}{12} &= \sin\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right) \\
&= \sin \frac{3\pi}{12} \cos \frac{2\pi}{12} + \cos \frac{3\pi}{12} \sin \frac{2\pi}{12} \\
\sin \frac{5\pi}{12} &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} \\
&= \frac{\sqrt{6} + \sqrt{2}}{4}
\end{aligned}$$

Example B

Given $\sin \alpha = \frac{12}{13}$, where α is in Quadrant II, and $\sin \beta = \frac{3}{5}$, where β is in Quadrant I, find the exact value of $\sin(\alpha + \beta)$.

Solution: To find the exact value of $\sin(\alpha+\beta)$, here we use $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. The values of $\sin\alpha$ and $\sin\beta$ are known, however the values of $\cos\alpha$ and $\cos\beta$ need to be found.

Use $\sin^2\alpha + \cos^2\alpha = 1$, to find the values of each of the missing cosine values.

For $\cos a : \sin^2\alpha + \cos^2\alpha = 1$, substituting $\sin\alpha = \frac{12}{13}$ transforms to $\left(\frac{12}{13}\right)^2 + \cos^2\alpha = \frac{144}{169} + \cos^2\alpha = 1$ or $\cos^2\alpha = \frac{25}{169} \cos\alpha = \pm\frac{5}{13}$, however, since α is in Quadrant II, the cosine is negative, $\cos\alpha = -\frac{5}{13}$.

For $\cos\beta$ use $\sin^2\beta+\cos^2\beta=1$ and substitute $\sin\beta=\frac{3}{5}, \left(\frac{3}{5}\right)^2+\cos^2\beta=\frac{9}{25}+\cos^2\beta=1$ or $\cos^2\beta=\frac{16}{25}$ and $\cos\beta=\pm\frac{4}{5}$ and since β is in Quadrant I, $\cos\beta=\frac{4}{5}$

Now the sum formula for the sine of two angles can be found:

$$\sin(\alpha+\beta)=\frac{12}{13}\times\frac{4}{5}+\left(-\frac{5}{13}\right)\times\frac{3}{5} \text{ or } \frac{48}{65}-\frac{15}{65}$$

$$\sin(\alpha+\beta)=\frac{33}{65}$$

Example C

Find the exact value of $\sin 15^\circ$

Solution: Recall that there are multiple angles that add or subtract to equal any angle. Choose whichever formula that you feel more comfortable with.

$$\begin{aligned}\sin 15^\circ &= \sin\left(45^\circ-30^\circ\right)\\ &= \sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ\\ \sin 15^\circ &= (.707)\times(.866)+(.707)\times(.5)\\ &= (.612262)\times(.3535)\\ &= .2164\end{aligned}$$

Vocabulary

Sine Sum Formula: The ***sine sum formula*** relates the sine of a sum of two arguments to a set of sine and cosines functions, each containing one argument.

Sine Difference Formula: The ***sine difference formula*** relates the sine of a difference of two arguments to a set of sine and cosines functions, each containing one argument.

Guided Practice

1. Find the exact value for $\sin 345^\circ$

2. Find the exact value for $\sin\frac{17\pi}{12}$

3. If $\sin y=-\frac{5}{13}$, y is in quad III, and $\sin z=\frac{4}{5}$, z is in quad II find $\sin(y+z)$

Solutions:

1.

$$\sin 345^\circ = \sin(300^\circ + 45^\circ) = \sin 300^\circ \cos 45^\circ + \cos 300^\circ \sin 45^\circ$$
$$= -\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4}$$

2.

$$\sin \frac{17\pi}{12} = \sin\left(\frac{9\pi}{12} + \frac{8\pi}{12}\right) = \sin\left(\frac{3\pi}{4} + \frac{2\pi}{3}\right) = \sin \frac{3\pi}{4} \cos \frac{2\pi}{3} + \cos \frac{3\pi}{4} \sin \frac{2\pi}{3}$$
$$= \frac{\sqrt{2}}{2} \cdot (-\frac{1}{2}) + -\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = -\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{-\sqrt{2} - \sqrt{6}}{4}$$

3.

If $\sin y = -\frac{5}{13}$ and in Quadrant III, then cosine is also negative. By the Pythagorean Theorem, the second leg is $12(5^2 + b^2 = 13^2)$, so $\cos y = -\frac{12}{13}$. If the $\sin z = \frac{4}{5}$ and in Quadrant II, then the cosine is also negative. By the Pythagorean Theorem, the second leg is $3(4^2 + b^2 = 5^2)$, so$\cos = -\frac{3}{5}$. To find $\sin(y + z)$, plug this information into the sine sum formula.

$$\sin(y + z) = \sin y \cos z + \cos y \sin z$$
$$= -\frac{5}{13} \cdot -\frac{3}{5} + -\frac{12}{13} \cdot \frac{4}{5} = \frac{15}{65} - \frac{48}{65} = -\frac{33}{65}$$

Concept Problem Solution

With the sine sum formula, you can break the sine into easier to calculate quantities:

$$\sin \frac{7\pi}{12} = \sin\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right)$$
$$= \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$
$$= \sin(\frac{\pi}{3}) \cos(\frac{\pi}{4}) + cos(\frac{\pi}{3}) sin(\frac{\pi}{4})$$
$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

Practice

Find the exact value for each sine expression.

1. $\sin 75°$
2. $\sin 105°$
3. $\sin 165°$
4. $\sin 255°$
5. $\sin -15°$

Write each expression as the sine of an angle.

6. $\sin 46° \cos 20° + \cos 46° \sin 20°$
7. $\sin 3x \cos 2x - \cos 3x \sin 2x$
8. $\sin 54° \cos 12° + \cos 54° \sin 12°$
9. $\sin 29° \cos 10° - \cos 29° \sin 10°$
10. $\sin 4y \cos 3y + \cos 4y \sin 2y$
11. Prove that $\sin(x - \frac{\pi}{2}) = -\cos(x)$
12. Suppose that x, y, and z are the three angles of a triangle. Prove that $\sin(x + y) = \sin(z)$
13. Prove that $\sin(\frac{\pi}{2} - x) = \cos(x)$
14. Prove that $\sin(x + \pi) = -\sin(x)$
15. Prove that $\sin(x - y) + \sin(x + y) = 2\sin(x)\cos(y)$