Tangent Sum and Difference Formulas

Suppose you were given two angles and asked to find the tangent of the difference of them. For example, can you compute:

$\tan(120^\circ - 40^\circ)$

Would you just subtract the angles and then take the tangent of the result? Or is something more complicated required to solve this problem? Keep reading, and by the end of this Concept, you'll be able to calculate trig functions like the one above.

Watch This

James Sousa: Sum and Difference Identities for Tangent

Guidance

In this Concept, we want to find a formula that will make computing the tangent of a sum of arguments or a difference of arguments easier. As first, it may seem that you should just add (or subtract) the arguments and take the tangent of the result. However, it's not quite that easy.

To find the sum formula for tangent:

$$\begin{aligned}
\tan(a+b) &= \frac{\sin(a+b)}{\cos(a+b)} && \text{Using } \tan\theta = \frac{\sin\theta}{\cos\theta} \\
&= \frac{\sin a\cos b + \sin b\cos a}{\cos a\cos b - \sin a\sin b} && \text{Substituting the sum formulas for sine and cosine} \\
&= \frac{\frac{\sin a\cos b+\sin b\cos a}{\cos a\cos b}}{\frac{\cos a\cos b-\sin a\sin b}{\cos a\cos b}} && \text{Divide both the numerator and the denominator by co} \\
&= \frac{\frac{\sin a\cos b}{\cos a\cos b} + \frac{\sin b\cos a}{\cos a\cos b}}{\frac{\cos a\cos b}{\cos a\cos b} - \frac{\sin a\sin b}{\cos a\cos b}} && \text{Reduce each of the fractions} \\
&= \frac{\frac{\sin a}{\cos a} + \frac{\sin b}{\cos b}}{1 - \frac{\sin a\sin b}{\cos a\cos b}} && \text{Substitute } \frac{\sin\theta}{\cos\theta} = \tan\theta \\
\tan(a+b) &= \frac{\tan a + \tan b}{1 - \tan a\tan b} && \text{Sum formula for tangent}
\end{aligned}$$

In conclusion, $\tan(a+b) = \frac{\tan a+\tan b}{1-\tan a\tan b}$. Substituting $-b$ for b in the above results in the difference formula for tangent:

$$\tan(a-b)=\frac{\tan a-\tan b}{1+\tan a\tan b}$$

Example A

Find the exact value of $\tan 285^\circ$.

Solution: Use the difference formula for tangent, with $285^\circ = 330^\circ - 45^\circ$

$$\begin{aligned}
\tan(330^\circ - 45^\circ) &= \frac{\tan 330^\circ - \tan 45^\circ}{1+\tan 330^\circ \tan 45^\circ}\\
&= \frac{-\frac{\sqrt{3}}{3}-1}{1-\frac{\sqrt{3}}{3}\cdot 1} = \frac{-3-\sqrt{3}}{3-\sqrt{3}}\\
&= \frac{-3-\sqrt{3}}{3-\sqrt{3}}\cdot\frac{3+\sqrt{3}}{3+\sqrt{3}}\\
&= \frac{-9-6\sqrt{3}-3}{9-3}\\
&= \frac{-12-6\sqrt{3}}{6}\\
&= -2-\sqrt{3}
\end{aligned}$$

To verify this on the calculator, $\tan 285^\circ = -3.732$ and $-2-\sqrt{3} = -3.732$.

Example B

Verify the tangent difference formula by finding $\tan\frac{6\pi}{6}$, since this should be equal to $\tan\pi = 0$.

Solution: Use the difference formula for tangent, with $\tan\frac{6\pi}{6} = \tan(\frac{7\pi}{6}-\frac{\pi}{6})$

$$\begin{aligned}
\tan(\frac{7\pi}{6}-\frac{\pi}{6}) &= \frac{\tan\frac{7\pi}{6}-\tan\frac{\pi}{6}}{1+\tan\frac{7\pi}{6}\tan\frac{\pi}{6}}\\
&= \frac{\frac{\sqrt{2}}{6}-\frac{\sqrt{2}}{6}}{1-\frac{\sqrt{2}}{6}\cdot\frac{\sqrt{2}}{6}} = \frac{0}{1-\frac{2}{36}}\\
&= \frac{0}{\frac{34}{36}}\\
&= 0
\end{aligned}$$

Example C

Find the exact value of $\tan 165^\circ$.

Solution: Use the difference formula for tangent, with $165^\circ = 225^\circ - 60^\circ$

$$\begin{aligned}\tan(225^\circ - 60^\circ) &= \frac{\tan 225^\circ - \tan 60^\circ}{1 + \tan 225^\circ \tan 60^\circ} \\ &= \frac{1 - \sqrt{3}}{1 - 1 \cdot \sqrt{3}} = 1\end{aligned}$$

Vocabulary

Tangent Sum Formula: The ***tangent sum formula*** relates the tangent of a sum of two arguments to a set of tangent functions, each containing one argument.

Tangent Difference Formula: The ***tangent difference formula*** relates the tangent of a difference of two arguments to a set of tangent functions, each containing one argument.

Guided Practice

1. Find the exact value for $\tan 75^\circ$

2. Simplify $\tan(\pi + \theta)$

3. Find the exact value for $\tan 15^\circ$

Solutions:

1.

$$\begin{aligned}\tan 75^\circ &= \tan(45^\circ + 30^\circ) \\ &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} \\ &= \frac{1 + \frac{\sqrt{3}}{3}}{1 - 1 \cdot \frac{\sqrt{3}}{3}} = \frac{\frac{3+\sqrt{3}}{3}}{\frac{3-\sqrt{3}}{3}} \\ &= \frac{3 + \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{3 + \sqrt{3}}{3 + \sqrt{3}} \\ &= \frac{9 + 6\sqrt{3} + 3}{9 - 3} = \frac{12 + 6\sqrt{3}}{6} \\ &= 2 + \sqrt{3}\end{aligned}$$

2. $\tan(\pi + \theta) = \frac{\tan \pi + \tan \theta}{1 - \tan \pi \tan \theta} = \frac{\tan \theta}{1} = \tan \theta$

3.

$$\begin{aligned}\tan 15^\circ &= \tan(45^\circ - 30^\circ)\\ &= \frac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ \tan 30^\circ}\\ &= \frac{1-\frac{\sqrt{3}}{3}}{1+1\cdot\frac{\sqrt{3}}{3}} = \frac{\frac{3-\sqrt{3}}{3}}{\frac{3+\sqrt{3}}{3}}\\ &= \frac{3-\sqrt{3}}{3+\sqrt{3}}\cdot\frac{3-\sqrt{3}}{3-\sqrt{3}}\\ &= \frac{9+6\sqrt{3}+3}{9-3} = \frac{12+6\sqrt{3}}{6}\\ &= 2+\sqrt{3}\end{aligned}$$

Concept Problem Solution

The Concept Problem asks you to find:

$\tan(120^\circ - 40^\circ)$

You can use the tangent difference formula:

$$\tan(a-b) = \frac{\tan a - \tan b}{1+\tan a \tan b}$$

to help solve this. Substituting in known quantities:

$$\tan(120^\circ - 40^\circ) = \frac{\tan 120^\circ - \tan 40^\circ}{1+(\tan 120^\circ)(\tan 40^\circ)} = \frac{-1.732-.839}{1+(-1.732)(.839)} = \frac{-2.571}{-.453148} = 5.674$$

Practice

Find the exact value for each tangent expression.

1. $\tan \frac{5\pi}{12}$
2. $\tan \frac{11\pi}{12}$
3. $\tan -165^\circ$
4. $\tan 255^\circ$
5. $\tan -15^\circ$

Write each expression as the tangent of an angle.

6. $\frac{\tan 15^\circ + \tan 42^\circ}{1-\tan 15^\circ \tan 42^\circ}$
7. $\frac{\tan 65^\circ - \tan 12^\circ}{1+\tan 65^\circ \tan 12^\circ}$
8. $\frac{\tan 10^\circ + \tan 50^\circ}{1-\tan 10^\circ \tan 50^\circ}$

9. $\frac{\tan 2y+\tan 4}{1-\tan 2 \tan 4y}$
10. $\frac{\tan x-\tan 3x}{1+\tan x \tan 3x}$
11. $\frac{\tan 2x-\tan y}{1+\tan 2x \tan y}$
12. Prove that $\tan(x+\frac{\pi}{4})=\frac{1+\tan(x)}{1-\tan(x)}$
13. Prove that $\tan(x-\frac{\pi}{2})=-\cot(x)$
14. Prove that $\tan(\frac{\pi}{2}-x)=\cot(x)$
15. Prove that $\tan(x+y)\tan(x-y)=\frac{\tan^2(x)-\tan^2(y)}{1-\tan^2(x)\tan^2(y)}$