# Lesson 4: Fundamental Theorem of Similarity (FTS)

Student Outcomes

- Students experimentally verify the properties related to the Fundamental Theorem of Similarity (FTS).

Lesson Notes

The goal of this activity is to show students the properties of the Fundamental Theorem of Similarity (FTS), in terms of dilation. FTS states that given a dilation from center O, and points P and Q (points O, P, Q are not collinear), the segments formed when you connect P to Q, and P' to Q', are parallel. More surprising is that $|P'Q'| = r|PQ|$. That is, the segment PQ, even though it was not dilated as points P and Q were, dilates to segment $P'Q'$ and the length of $P'Q'$ is the length of PQ multiplied by the scale factor. The picture that follows is what the end product of the activity should look like. Also, consider showing the diagram (without the lengths of segments), and ask students to make conjectures about the relationships between the lengths of segments PQ and $P'Q'$.
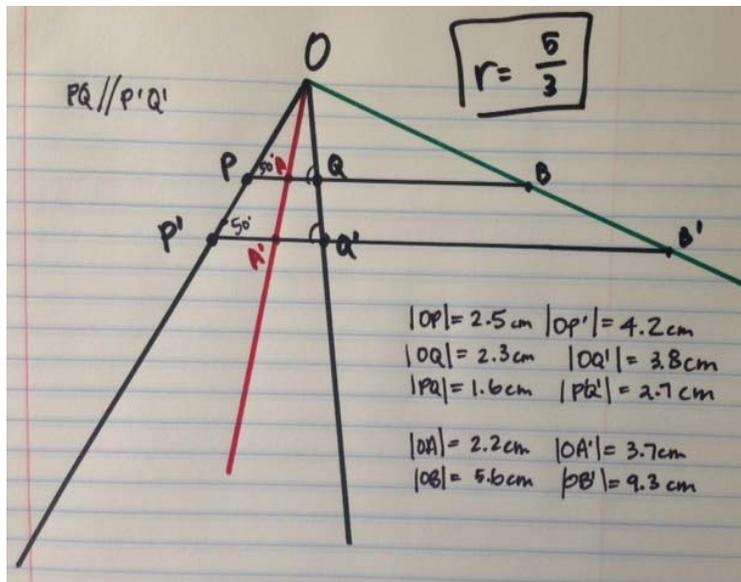

Classwork

Discussion (30 minutes)

For this discussion, students will need a piece of lined paper, a cm ruler, a protractor, and a four-function (or scientific) calculator.

- The last few days have focused on dilation. We now want to use what we know about dilation to come to some conclusions about the concept of similarity in general.

- A regular piece of notebook paper can be a great tool for discussing similarity. What do you notice about the lines on the notebook paper?

    □ *The lines on the notebook paper are parallel, that is, they never intersect.*

- Keep that information in mind as we proceed through this activity. On the first line of your paper, mark a point O. We will use this as our center.

- From point O, draw a ray $\overrightarrow{OP}$. Mark the point P a few lines down from the center. Now, choose a P' farther down the ray, also on one of the lines of the notebook paper. For example, you may have placed point P, 3 lines down from the center, and point P', 5 lines down from the center.

- Use the definition of dilation to describe the lengths along this ray.

    □ *By definition of dilation, $|OP'| = r|OP|$.*

- Recall that we can calculate the scale factor using the following computation: $\dfrac{|OP'|}{|OP|} = r$. In my example, the scale factor $r = \dfrac{5}{3}$ because OP' is 5 lines from the center, and OP is 3 lines down. On the top of your paper, write down the scale factor that you have used.

- Now draw another ray, $\overrightarrow{OQ}$. Use the same scale factor to mark points Q and Q'. In my example, I would place Q, 3 lines down, and Q', 5 lines down from the center.

- Now connect point P to point Q, and point P' to point Q'. What do you notice about lines PQ and $P'Q'$?

    □ The lines PQ and $P'Q'$ fall on the notebook lines, which means that PQ and $P'Q'$ are parallel lines.

- Use your protractor to measure angles $\angle OPQ$ and $\angle OP'Q'$. What do you notice and why is it so?

    □ *Angles $\angle OPQ$ and $\angle OP'Q'$ are equal in measure. They must be equal in measure because they are corresponding angles of parallel lines (PQ and P'Q') cut by a transversal (ray $\overrightarrow{OP}$).*

- (Consider asking students to write their answers to the following question in their notebooks and to justify their answers.) Now, without using your protractor, what can you say about angles $\angle OQP$ and $\angle OQ'P'$?

    □ *These angles are also equal for the same reason; they are corresponding angles of parallel lines (PQ and P'Q') cut by a transversal (ray $\overrightarrow{OQ}$).*

- Use your cm ruler to measure the lengths OP and OP'. By definition of dilation, we expect $|OP'| = r|OP|$ (that is, we expect the length of OP' to be equal to the scale factor times the length of OP). Verify that this is true. Do the same for lengths OQ and OQ'.

    □ *Sample of what student work may look like:*
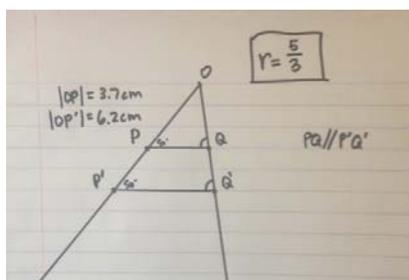
Note to Teacher:

A cm ruler will be easier for students to come up with a precise measurement. Also, let students know that it is okay if their measurements are off by a tenth of a cm because that difference can be attributed to human error.

- Bearing in mind that we have dilated from center O, points P and Q along their respective rays. Do you expect the segments PQ and $P'Q'$ to have the relationship $|P'Q'| = r|PQ|$?
  - *(Some students may say yes. If they do, ask for a convincing argument. At this point they have knowledge of dilating segments, but that is not what we have done here. We have dilated points and then connected them to draw the segments.)*
- Measure the segments PQ and $P'Q'$ to see if they have the relationship $|P'Q'| = r|PQ|$.
- It should be somewhat surprising that in fact, segments PQ and $P'Q'$ enjoy the same properties as the segments that we actually dilated.
- Now mark a point A on line PQ, between points P and Q. Draw a ray from center O through point A and then mark A' on the line $P'Q'$. Do you think $|P'A'| = r|PA|$? Measure the segments and use your calculator to check.
  - *Students should notice that these new segments also have the same properties as the dilated segments.*
- Now, mark a point B on the line PQ, but this time not on the segment PQ (i.e., not between points P and Q). Again, draw the ray from center O through point B and mark the point B' on the line $P'Q'$. Select any segment, $AB, PB, QB,$ and verify that it has the same property as the others.
  - *Sample of what student work may look like:*
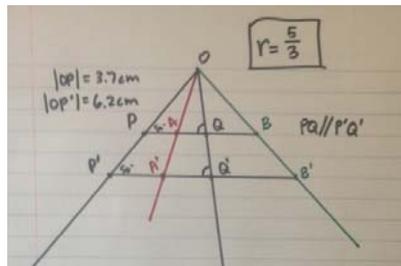
- Will this always happen, no matter the scale factor or placement of points $P, Q, A,$ and B?
  - *Yes, I believe this is true. One main reason is that everyone in class probably picked different points and I'm sure many of us used different scale factors.*
- Describe the rule or pattern that we have discovered in your own words.

MP.8 Encourage students to write and collaborate with a partner to answer this question. Once students have finished their work, lead a discussion that crystallizes the information in the theorem that follows.

- We have just experimentally verified the properties of the Fundamental Theorem of Similarity (FTS) in terms of dilation. Namely, that the parallel line segments connecting dilated points are related by the same scale factor as the segments that are dilated.

  Theorem: Given a dilation with center O and scale factor r, then for any two points P and Q in the plane so that O, P, and Q are not collinear, the lines PQ and $P'Q'$ are parallel, where $P' = dilation(P)$ and $Q' = dilation(Q)$, and furthermore, $|P'Q'| = r|PQ|$.

Ask students to paraphrase the theorem in their own words or offer them the following version of the theorem: FTS states that given a dilation from center O, and points P and Q (points O, P, Q are not on the same line), the segments formed when you connect P to Q, and P' to Q', are parallel. More surprising is the fact that the segment PQ, even though it was not dilated as points P and Q were, dilates to segment $P'Q'$ and the length of $P'Q'$ is the length of PQ multiplied by the scale factor.

- Now that we are more familiar with properties of dilations and similarity, we will begin using these properties in the next few lessons to do things like verify similarity of figures.

Exercise (5 minutes)

Exercise

1.   In the diagram below, points R and S have been dilated from center O, by a scale factor of $r = 3$.
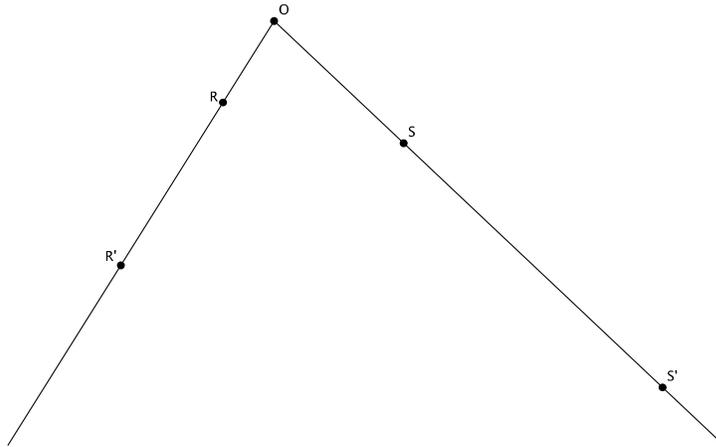
a.   If the length of $|OR| = 2.3$ cm, what is the length of $|OR'|$?

   $|OR'| = 3(2.3) = 6.9$ *cm*

b.   If the length of $|OS| = 3.5$ cm, what is the length of $|OS'|$?

   $|OS'| = 3(3.5) = 10.5$ *cm*

c.   Connect the point R to the point S and the point R' to the point S'. What do you know about lines RS and $R'S'$?

   The lines RS and R'S' are parallel.

d.   What is the relationship between the length of RS and the length of $R'S'$?

   The length of R'S' will be equal to the length of RS, times the scale factor of 3 (i.e., $|R'S'| = 3|RS|$).

e.   Identify pairs of angles that are equal in measure. How do you know they are equal?

   $\angle ORS = \angle OR'S'$ and $\angle OSR = \angle OS'R'$. They are equal because they are corresponding angles of parallel lines cut by a transversal.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that the following is true: If $|OP'| = r|OP|$ and $|OQ'| = r|OQ|$, then $|P'Q'| = r|PQ|$. In other words, under a dilation from a center with scale factor r, a segment multiplied by the scale factor results in the length of the dilated segment.

- We also know that the lines PQ and $P'Q'$ are parallel.

- We verified the Fundamental Theorem of Similarity in terms of dilation using an experiment with notebook paper.

Lesson Summary

Theorem: Given a dilation with center O and scale factor r, then for any two points P and Q in the plane so that O, P, and Q are not collinear, the lines PQ and $P'Q'$ are parallel, where $P' = dilation(P)$ and $Q' = dilation(Q)$, and furthermore, $|P'Q'| = r|PQ|$.

Exit Ticket (5 minutes)

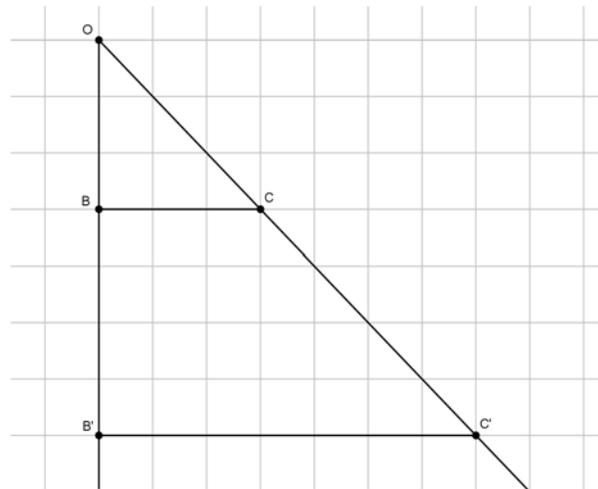Name _____   Date_____

Lesson 4: Fundamental Theorem of Similarity (FTS)

Exit Ticket

Steven sketched the following diagram on graph paper. He dilated points B and C from point O. Answer the following questions based on his drawing:

1.   What is the scale factor r? Show your work.

2.   Verify the scale factor with a different set of segments.

3.   Which segments are parallel? How do you know?

4.   Are $\angle OBC = \angle OB'C'$ right angles? How do you know?

Exit Ticket Sample Solutions

> 1. **What is the scale factor r? Show your work.**
>
> $|OB'| = r|OB|$
> $7 = r \times 3$
> $\dfrac{7}{3} = r$
>
> 2. **Verify the scale factor with a different set of segments.**
>
> $|B'C'| = r|BC|$
> $7 = r \times 3$
> $\dfrac{7}{3} = r$
>
> 3. **Which segments are parallel? How do you know?**
>
>   *Segments BC and $B'C'$ are parallel since they lie on the grid lines of the paper, which are parallel.*
>
> 4. **Are $\angle OBC = \angle OB'C'$ right angles? How do you know?**
>
>   *The grid lines on graph paper are perpendicular, and since perpendicular lines form right angles, $\angle OBC = \angle OB'C'$ are right angles.*

Problem Set Sample Solutions

Students verify that the Fundamental Theorem of Similarity holds true when the scale factor r is $0 < r < 1$.

> 1. **Use a piece of notebook paper to verify the Fundamental Theorem of Similarity for a scale factor r that is $0 < r < 1$.**
>
>   ✓ **Mark a point O on the first line of notebook paper.**
>
>   ✓ **Draw a ray, $\overrightarrow{OP}$. Mark the point P on a line, several lines down from the center. Mark the point P' on the ray, and on a line of the notebook paper, closer to O than you placed point P. This ensures that you have a scale factor that is $0 < r < 1$. Write your scale factor at the top of the notebook paper.**
>
>   ✓ **Draw another ray, $\overrightarrow{OQ}$, and mark the points Q and Q' according to your scale factor.**
>
>   ✓ **Connect points P and Q. Then, connect points P' and Q'.**
>
>   ✓ **Place a point A on line PQ between points P and Q. Draw ray $\overrightarrow{OA}$. Mark the point A' at the intersection of line $P'Q'$ and ray $\overrightarrow{OA}$.**

Sample student work shown in the picture below:
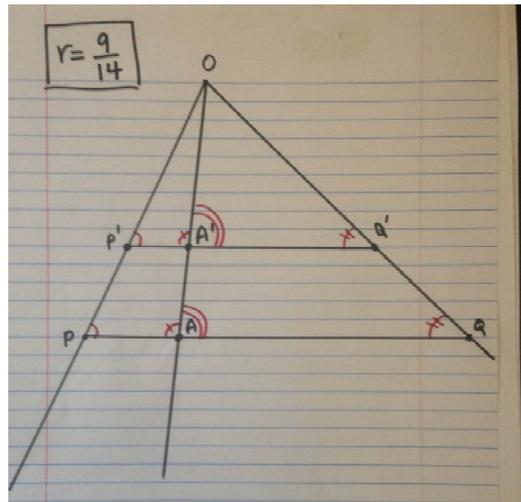
a. Are lines PQ and $P'Q'$ parallel lines? How do you know?

    Yes, the lines PQ and $P'Q'$ are parallel. The notebook lines are parallel and these lines fall on the notebook lines.

b. Which, if any, of the following pairs of angles are equal? Explain.
    i. $\angle OPQ$ and $\angle OP'Q'$
    ii. $\angle OAQ$ and $\angle OA'Q'$
    iii. $\angle OAP$ and $\angle OA'P'$
    iv. $\angle OQP$ and $\angle OQ'P'$

    All four pairs of angles are equal because each pair of angles are corresponding angles of parallel lines cut by a transversal. In each case, the parallel lines are PQ and $P'Q'$ and the transversal is their respective ray.

c. Which, if any, of the following statements are true? Show your work to verify or dispute each statement.
    i. $|OP'| = r|OP|$
    ii. $|OQ'| = r|OQ|$
    iii. $|P'A'| = r|PA|$
    iv. $|A'Q'| = r|AQ|$

    All four of the statements are true. Verify that students have shown that the length of the dilated segment was equal to the scale factor multiplied by the original segment length.

d. Do you believe that the Fundamental Theorem of Similarity (FTS) is true even when the scale factor is $0 < r < 1$. Explain.

    Yes, because I just experimentally verified the properties of FTS for when the scale factor is $0 < r < 1$.

2.   Caleb sketched the following diagram on graph paper. He dilated points B and C from center O.
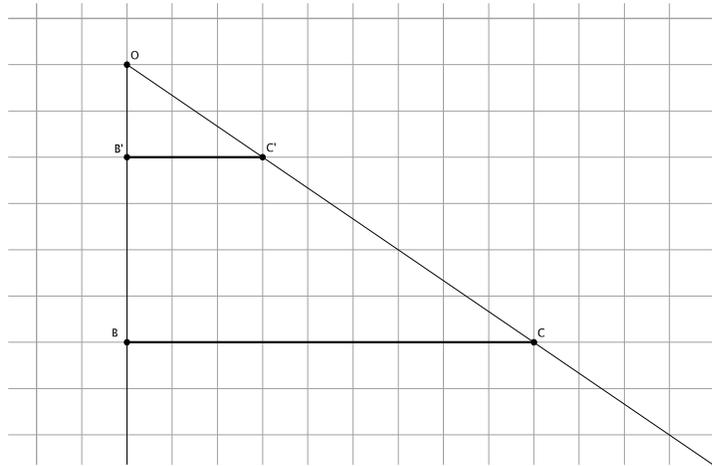
a.   **What is the scale factor r? Show your work.**

   $|OB'| = r|OB|$

   $\quad 2 = r \times 6$

   $\quad \dfrac{2}{6} = r$

   $\quad \dfrac{1}{3} = r$

b.   **Verify the scale factor with a different set of segments.**

   $|B'C'| = r|BC|$

   $\quad 3 = r \times 9$

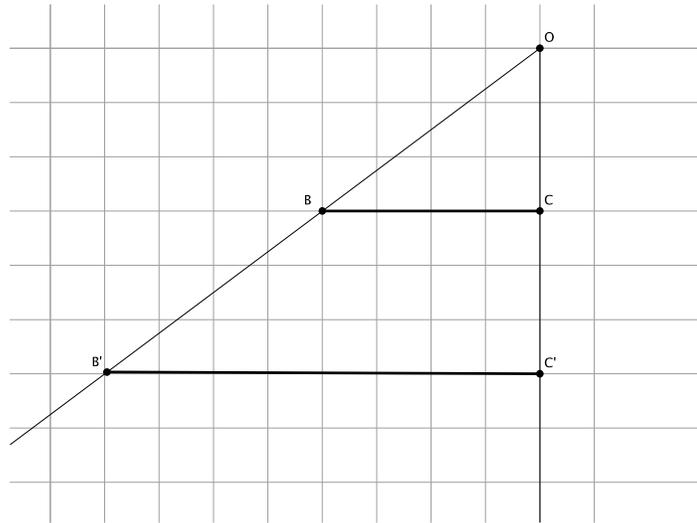 $\quad \dfrac{3}{9} = r$

   $\quad \dfrac{1}{3} = r$

c.   **Which segments are parallel? How do you know?**

   Segment BC and $B'C'$ are parallel. They lie on the lines of the graph paper, which are parallel.

d.   **Which angles are equal in measure? How do you know?**

   $\angle OB'C' = \angle OBC$, and $\angle OC'B' = \angle OCB$ because they are corresponding angles of parallel lines cut by a transversal.

3.   Points B and C were dilated from center O.

a.   What is the scale factor r? Show your work.

$$|OC'| = r|OC|$$
$$6 = r \times 3$$
$$\frac{6}{3} = r$$
$$2 = r$$

b.   If the length of $|OB| = 5$, what is the length of $|OB'|$?

$$|OB'| = r|OB|$$
$$|OB'| = 2 \times 5$$
$$|OB'| = 10$$

c.   How does the perimeter of triangle OBC compare to the perimeter of triangle $OB'C'$?

The perimeter of triangle OBC is 12 units and the perimeter of triangle $OB'C'$ is 24 units.

d.   Did the perimeter of triangle $OB'C' = r \times$ (perimeter of triangle OBC)? Explain.

Yes, the perimeter of triangle $OB'C'$ was twice the perimeter of triangle OBC, which makes sense because the dilation increased the length of each segment by a scale factor of 2. That means that each side of triangle $OB'C'$ was twice as long as each side of triangle OBC.