Lesson 13: Proof of the Pythagorean Theorem

Student Outcomes

- Students practice applying the Pythagorean Theorem to find lengths of right triangles in two dimensions.

Lesson Notes

Since 8.G.6 and 8.G.7 are post-test standards, this lesson is designated as an extension lesson for this module. However, the content within this lesson is prerequisite knowledge for Module 7. If this lesson is not used with students as part of the work within Module 3, it must be used with students prior to beginning work on Module 7. Please realize that many mathematicians agree that the Pythagorean Theorem is the most important theorem in geometry and has immense implications in much of high school mathematics in general (e.g., learning of quadratics, trigonometry, etc.). It is crucial that students see the teacher explain several proofs of the Pythagorean Theorem and practice using it before being expected to produce a proof on their own.

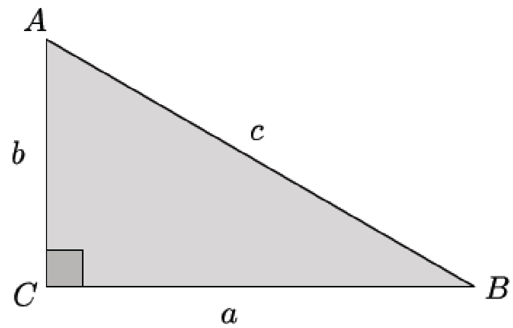
Classwork

Discussion (20 minutes)

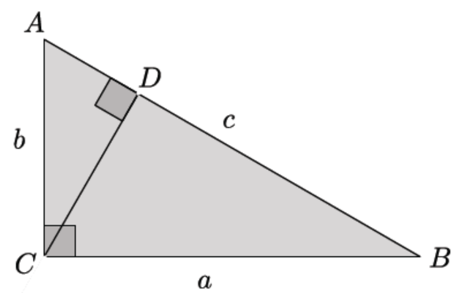
The following proof of the Pythagorean Theorem is based on the fact that similarity is transitive. It begins with the right triangle, shown on the next page, split into two other right triangles. The three triangles are placed in the same orientation, and students verify that one pair of triangles are similar using the AA criterion, then a second pair of triangles are shown to be similar using the AA criterion, and then finally all three triangles are shown to be similar by the fact that similarity is transitive. Once it is shown that all three triangles are in fact similar, the theorem is proved by comparing the ratios of corresponding side lengths. Because some of the triangles share side lengths that are the same (or sums of lengths), then the formula $a^2 + b^2 = c^2$ is derived. Symbolic notation is used explicitly for the lengths of sides. For that reason, it may be beneficial to do this proof simultaneously with triangles that have concrete numbers for side lengths. Another option to prepare students for the proof is showing the video presentation first, then working through this Socratic discussion.

- The concept of similarity can be used to prove one of the great theorems in mathematics, the Pythagorean Theorem. What do you recall about the Pythagorean Theorem from our previous work?
  - *The Pythagorean Theorem is a theorem about the lengths of the legs and the hypotenuse of right triangles. Specifically, if a and b are the legs of a right triangle and c is the hypotenuse, then $a^2 + b^2 = c^2$. The hypotenuse is the longest side of the triangle, and it is opposite the right angle.*

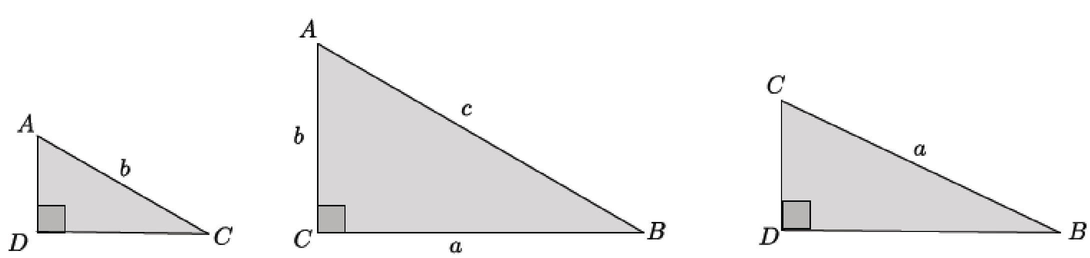
- What we are going to do in this lesson is take a right triangle, $\triangle ABC$, and use what we know about similarity of triangles to prove $a^2 + b^2 = c^2$.



- For the proof, we will draw a line from vertex C to a point D so that the line is perpendicular to side AB.



- We draw this particular line, line CD, because it divides the original triangle into three similar triangles. Before we move on, can you name the three triangles?
  - *The three triangles are* $\triangle ABC$, $\triangle ACD$, *and* $\triangle BCD$.
- Let's look at the triangles in a different orientation in order to see why they are similar. We can use our basic rigid motions to separate the three triangles. Doing so ensures that the lengths of segments and degrees of angles are preserved.



- In order to have similar triangles, they must have two common angles, by the AA criterion. Which angles prove that $\triangle ADC$ and $\triangle ACB$ similar?
  - *It is true that* $\triangle ADC \sim \triangle ACB$ *because they each have a right angle, and they each share* $\angle A$.
- What must that mean about $\angle C$ from $\triangle ADC$ and $\angle B$ from $\triangle ACB$?
  - *It means that the angles correspond and must be equal in measure because of the triangle sum theorem.*

- Which angles prove that $\triangle ACB$ and $\triangle CDB$ similar?
  - *It is true that $\triangle ACB \backsim \triangle CDB$ because they each have a right angle and they each share $\angle B$.*
- What must that mean about $\angle A$ from $\triangle ACB$ and $\angle C$ from $\triangle CDB$?
  - *The angles correspond and must be equal in measure because of the triangle sum theorem.*
- If $\triangle ADC \backsim \triangle ACB$ and $\triangle ACB \backsim \triangle CDB$, is it true that $\triangle ADC \backsim \triangle CDB$? How do you know?
  - *Yes, because similarity is a transitive relation.*
- When we have similar triangles, we know that their side lengths are proportional. Therefore, if we consider $\triangle ADC$ and $\triangle ACB$, we can write

$$\frac{|AC|}{|AB|} = \frac{|AD|}{|AC|}.$$

By the cross-multiplication algorithm,

$$|AC|^2 = |AB| \cdot |AD|.$$

By considering $\triangle ACB$ and $\triangle CDB$, we can write

$$\frac{|BA|}{|BC|} = \frac{|BC|}{|BD|}.$$

Which again by the cross-multiplication algorithm,

$$|BC|^2 = |BA| \cdot |BD|.$$

If we add the two equations together, we get

$$|AC|^2 + |BC|^2 = |AB| \cdot |AD| + |BA| \cdot |BD|.$$

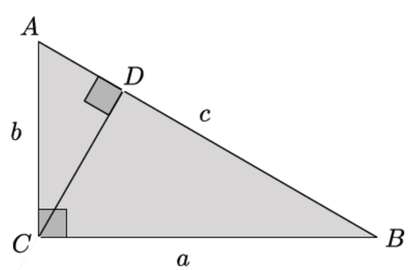
By the distributive property, we can rewrite the right side of the equation because there is a common factor of $|AB|$. Now we have

$$|AC|^2 + |BC|^2 = |AB|(|AD| + |BD|).$$

Keeping our goal in mind, we want to prove that $a^2 + b^2 = c^2$; *let's see how close we are.*

> *Scaffolding:*
> Use concrete numbers to quickly convince students that adding two equations together leads to another true equation. For example: $5 = 3 + 2$ and $8 = 4 + 4$, therefore, $5 + 8 = 3 + 2 + 4 + 4$.

- Using our diagram where three triangles are within one, (shown below), what side lengths are represented by $|AC|^2 + |BC|^2$?



  - *AC is side length b, and BC is side length a, so the left side of our equation represents $a^2 + b^2$.*
- Now let's examine the right side of our equation: $|AB|(|AD| + |BD|)$. We want this to be equal to c^2; does it?
  - *If we add the lengths AD and BD we get the entire length of AB; therefore, we have $|AB|(|AD| + |BD|) = |AB| \cdot |AB| = |AB|^2 = c^2$.*
- We have just proven the Pythagorean Theorem using what we learned about similarity. At this point we have seen the proof of the theorem in terms of congruence and now similarity.

Video Presentation (7 minutes)

The video located at the following link is an animation[1] of the preceding proof using similar triangles: http://www.youtube.com/watch?v=QCyvxYLFSfU

Exercises 1–3 (8 minutes)

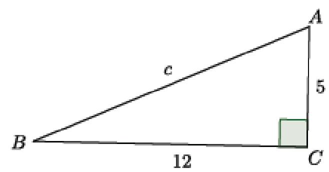
Students work independently to complete Exercises 1–3.

Exercises

Use the Pythagorean Theorem to determine the unknown length of the right triangle.

1. **Determine the length of side c in each of the triangles below.**

   a.



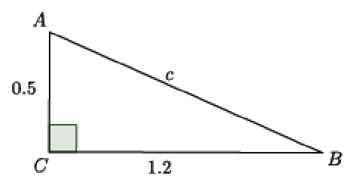
$5^2 + 12^2 = c^2$

$25 + 144 = c^2$

$169 = c^2$

$13 = c$

   b.



$0.5^2 + 1.2^2 = c^2$

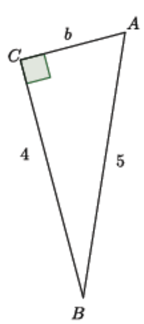
$0.25 + 1.44 = c^2$

$1.69 = c^2$

$1.3 = c$

2. **Determine the length of side b in each of the triangles below.**

   a.



$4^2 + b^2 = 5^2$

$16 + b^2 = 25$

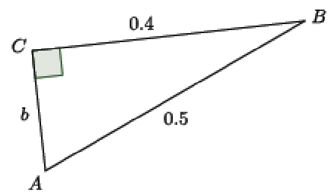
$16 - 16 + b^2 = 25 - 16$

$b^2 = 9$

$b = 3$

[1] Animation developed by Larry Francis.

b.



$0.4^2 + b^2 = 0.5^2$

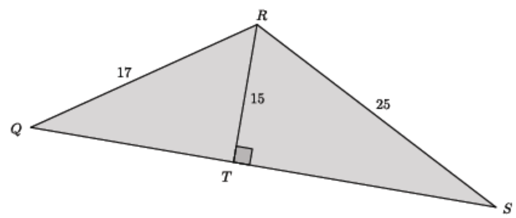
$1.6 + b^2 = 2.5$

$1.6 - 1.6 + b^2 = 2.5 - 1.6$

$b^2 = 0.9$

$b = 0.3$

3. Determine the length of QS. (Hint: Use the Pythagorean Theorem twice.)



$15^2 + |QT|^2 = 17^2$

$225 + |QT|^2 = 289$

$225 - 225 + |QT|^2 = 289 - 225$

$|QT|^2 = 64$

$|QT| = 8$

$15^2 + |TS|^2 = 25^2$

$225 + |TS|^2 = 625$

$225 - 225 + |TS|^2 = 625 - 225$

$|TS|^2 = 400$

$|TS| = 20$

Since $|QT| + |TS| = |QS|$, then the length of QS is $8 + 20$, which is 28.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We have now seen another proof of the Pythagorean Theorem, but this time we used what we knew about similarity, specifically similar triangles.
- We practiced using the Pythagorean Theorem to find unknown lengths of right triangles.

Exit Ticket (5 minutes)

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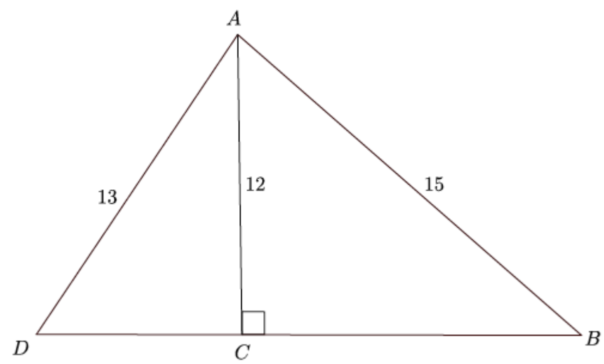
Lesson 13: Proof of the Pythagorean Theorem

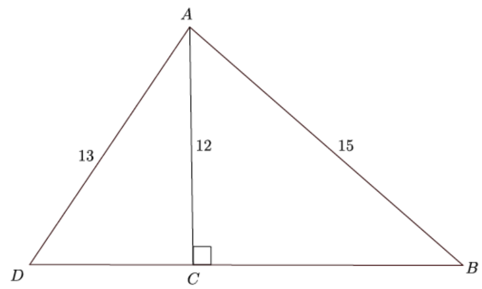
Exit Ticket

Determine the length of side BD in the triangle below.



Exit Ticket Sample Solutions

Determine the length of BD in the triangle below.



First determine the length of side BC.

$$12^2 + BC^2 = 15^2$$
$$144 + BC^2 = 225$$
$$BC^2 = 225 - 144$$
$$BC^2 = 81$$
$$BC = 9$$

Then determine the length of side CD.

$$12^2 + CD^2 = 13^2$$
$$144 + CD^2 = 169$$
$$CD^2 = 169 - 144$$
$$CD^2 = 25$$
$$CD = 5$$

Adding the length of BC and CD will determine the length of BD; therefore, $5 + 9 = 14$. BD has a length of 14.

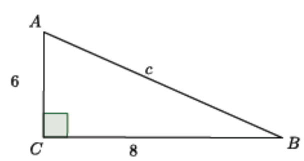
Problem Set Sample Solutions

Students practice using the Pythagorean Theorem to find unknown lengths of right triangles.

Use the Pythagorean theorem to determine the unknown length of the right triangle.

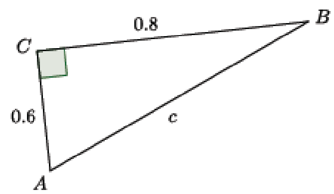
1. **Determine the length of side c in each of the triangles below.**

   a.



$$6^2 + 8^2 = c^2$$
$$36 + 64 = c^2$$
$$100 = c^2$$
$$10 = c$$

b.



$0.6^2 + 0.8^2 = c^2$

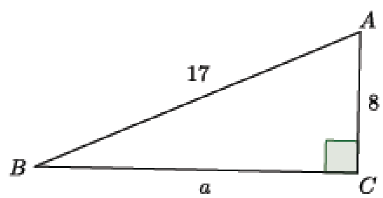
$0.36 + 0.64 = c^2$

$1 = c^2$

$1 = c$

2. **Determine the length of side a in each of the triangles below.**

a.



$a^2 + 8^2 = 17^2$

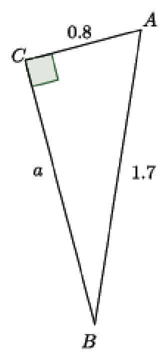
$a^2 + 64 = 289$

$a^2 + 64 - 64 = 289 - 64$

$a^2 = 225$

$a = 15$

b.



$a^2 + 0.8^2 = 1.7^2$

$a^2 + 0.64 = 2.89$

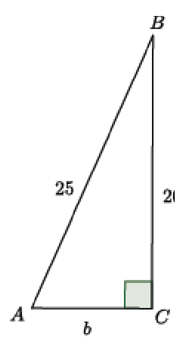
$a^2 + 0.64 - 0.64 = 2.89 - 0.64$

$a^2 = 2.25$

$a = 1.5$

3. **Determine the length of side b in each of the triangles below.**

a.



$20^2 + b^2 = 25^2$

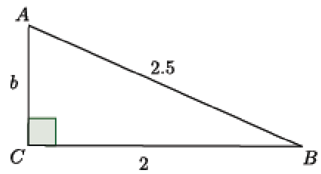
$400 + b^2 = 625$

$400 - 400 + b^2 = 625 - 400$

$b^2 = 225$

$b = 15$

b.



$2^2 + b^2 = 2.5^2$

$4 + b^2 = 6.25$

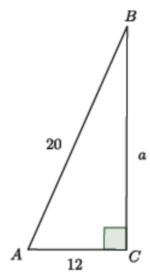
$4 - 4 + b^2 = 6.25 - 4$

$b^2 = 2.25$

$b = 1.5$

4. **Determine the length of side a in each of the triangles below.**

a.



$a^2 + 12^2 = 20^2$

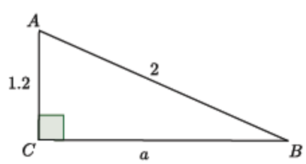
$a^2 + 144 = 400$

$a^2 + 144 - 144 = 400 - 144$

$a^2 = 256$

$a = 16$

b.



$a^2 + 1.2^2 = 2^2$

$a^2 + 1.44 = 4$

$a^2 + 1.44 - 1.44 = 4 - 1.44$

$a^2 = 2.56$

$a = 1.6$

5. **What did you notice in each of the pairs of problems 1–4? How might what you noticed be helpful in solving problems like these?**

In each pair of problems, the problems and solutions were similar. For example, in problem 1, part (a) showed the sides of the triangle were $6, 8,$ *and* 10*, and in part (b) they were* 0.6*,* 0.8*, and* 1*. The side lengths in part (b) were a tenth of the value of the lengths in part (a). The same could be said about parts (a) and (b) of problems 2–4. This might be helpful for solving problems in the future. If I'm given sides lengths that are decimals, then I could multiply them by a factor of* 10 *to make whole numbers, which are easier to work with. Also, if I know common numbers of Pythagorean Theorem, like side lengths of* $3, 4,$ *and* 5*, then I will recognize them more easily in their decimal forms, i.e.,* $0.3, 0.4,$ *and* 0.5*.*