Lesson 15: The Slope of a Non-Vertical Line

Student Outcomes

- Students know slope is a number that describes the steepness or slant of a line.
- Students interpret the unit rate as the slope of a graph.

Lesson Notes

In Lesson 13, we made some predictions about what the graph of a linear equation would look like. In all cases, we predicted that the graph of a linear equation in two variables would be a line. In Lesson 14, we learned that the graph of the linear equation $x = c$ is the vertical line passing through the point $(c, 0)$ and the graph of the linear equation $y = c$ is the horizontal line passing through the point $(0, c)$.

We would like to prove that our predictions are true: That the graph of a linear equation in two variables is a line. Before we do, we need some tools:

1. We must define a number for each non-vertical line that can be used to measure the steepness or slant of the line. Once defined, this number will be called the *slope* of the line and is often referred to as the *rate of change*.

Rate of change is terminology that will be used in later lessons. In this first exposure to slope, it is referred to as steepness or slant of a line. In this lesson, students make observations about the steepness of a line. Further, students give directions about how to get from one point on the line to another point on the line. This leads students to the conclusion that the units in the directions have the same ratio. Students then compare ratios between graphs and describe lines as steeper or flatter.

2. We must show that *any* two points on a non-vertical line can be used to find the slope of the line.
3. We must show that the line joining two points on a graph of a linear equation of the form $y = mx + b$ has the slope m.
4. We must show that there is only one line passing through a given point with a given slope.

These tools will be developed over the next few lessons. It is recommended that students are made aware of the four-part plan to achieve the goal of proving that the graph of a linear equation in two variables is a line as these parts are referenced in the next few lessons to help students make sense of the problem and persevere in solving it. In this lesson, we will look specifically at what is meant by the terms steepness and slant by defining the slope of a line. This lesson defines slope in the familiar context of unit rate; that is, slope is defined when the horizontal distance between two points is fixed at one. Defining slope this way solidifies the understanding that the unit rate is the slope of a graph. Further, students see that the number that describes slope is the distance between the y-coordinates leading to the general slope formula.

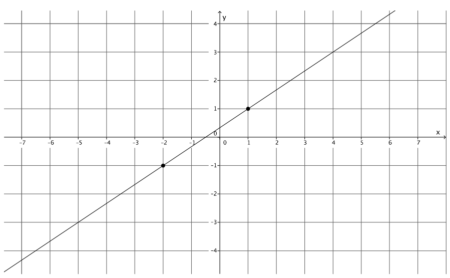
Opening (8 minutes)

To develop conceptual understanding of slope, have students complete the opening activity where they make informal observations about the steepness of lines. Model for students how to answer the questions with the first pair of graphs. Then have students work independently or in pairs to describe how to move from one point to another on a line in terms of units up or down and units right or left. Students also compare the ratios of their descriptions and relate the ratios to the steepness or flatness of the graph of the line.

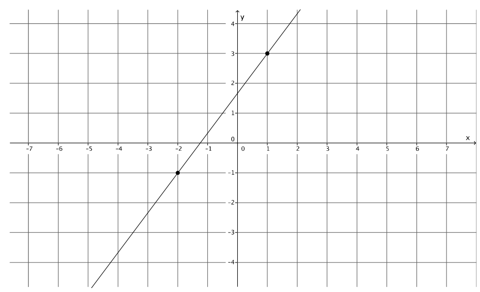
Examine each pair of graphs and answer the questions that follow.

Example

Graph A



Graph B



a. **Which graph is steeper?**

It looks like Graph B is steeper.

b. **Write directions that explain how to move from one point on the graph to the other for each of Graph A and Graph B.**

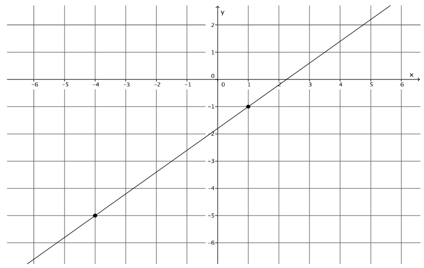
For Graph A, move 2 units up and 3 units right. For Graph B, move 4 units up and 3 units right.

c. **Write the directions from part (b) as ratios, and then compare the ratios. How does this relate to which graph was steeper in part (a)?**

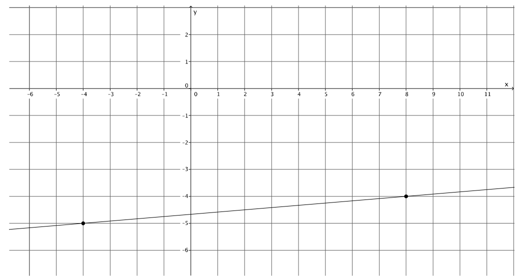
$\frac{2}{3} < \frac{4}{3}$ *Graph B was steeper and had the greater ratio.*

Pair 1:

Graph A



Graph B



a. **Which graph is steeper?**

It looks like Graph A is steeper.

b. **Write directions that explain how to move from one point on the graph to the other for each of Graph A and Graph B.**

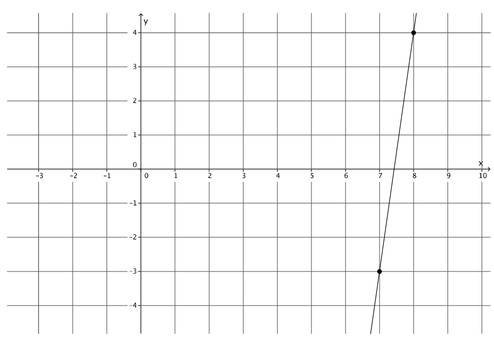
For Graph A, move 4 units up and 5 units right. For Graph B, move 1 unit up and 12 units right.

c. **Write the directions from part (b) as ratios, and then compare the ratios. How does this relate to which graph was steeper in part (a)?**

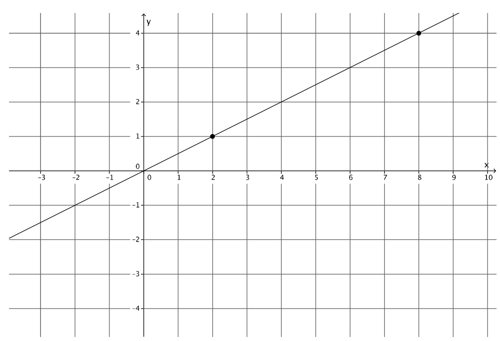
$\frac{4}{5} > \frac{1}{12}$ *Graph A was steeper and had the greater ratio.*

Pair 2:

Graph A



Graph B



a. **Which graph is steeper?**

It looks like Graph A is steeper.

b. **Write directions that explain how to move from one point on the graph to the other for each of Graph A and Graph B.**

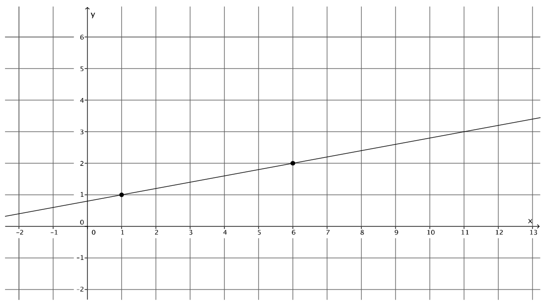
For Graph A, move 7 units up and 1 unit right. For Graph B, move 3 units up and 6 units right.

c. **Write the directions from part (b) as ratios, and then compare the ratios. How does this relate to which graph was steeper in part (a)?**

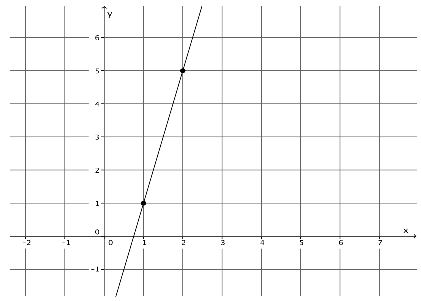
$\frac{7}{1} > \frac{3}{6}$ *Graph A was steeper and had the greater ratio.*

Pair 3:

Graph A



Graph B



a. **Which graph is steeper?**

It looks like Graph B is steeper.

b. **Write directions that explain how to move from one point on the graph to the other for each of Graph A and Graph B.**

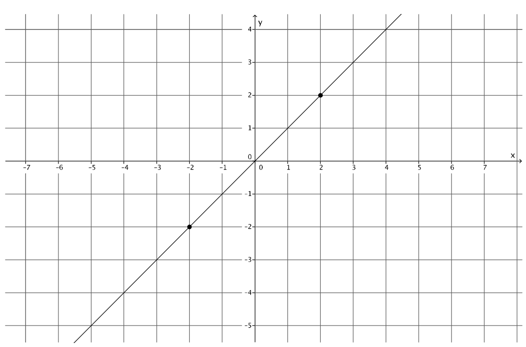
For Graph A, move 1 unit up and 5 units right. For Graph B, move 4 units up and 1 unit right.

c. **Write the directions from part (b) as ratios, and then compare the ratios. How does this relate to which graph was steeper in part (a)?**

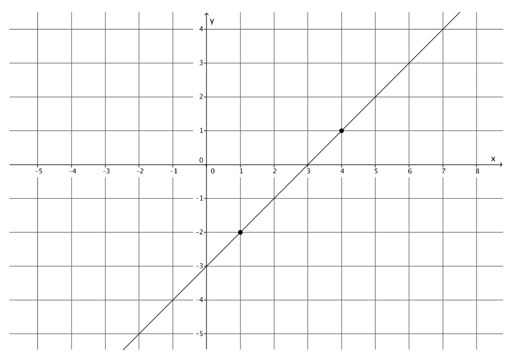
$\frac{1}{5} < \frac{4}{1}$ *Graph B was steeper and had the greater ratio.*

Pair 4:

Graph A



Graph B



a. **Which graph is steeper?**

They look about the same steepness.

b. **Write directions that explain how to move from one point on the graph to the other for each of Graph A and Graph B.**

For Graph A, move 4 units up and 4 units right. For Graph B, move 3 units up and 3 units right.

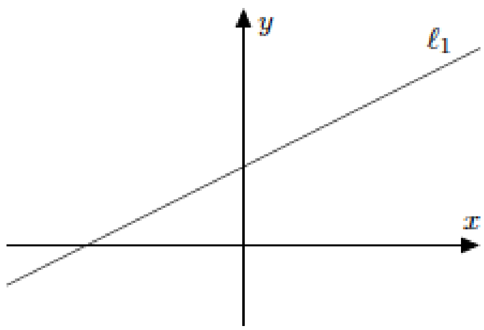
c. **Write the directions from part (b) as ratios, and then compare the ratios. How does this relate to which graph was steeper in part (a)?**

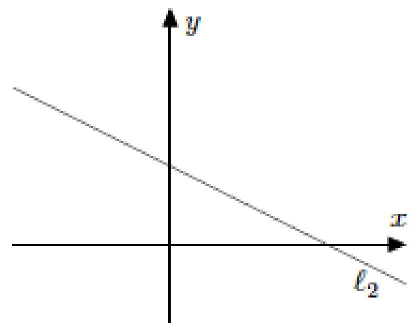
$\frac{4}{4} = \frac{3}{3}$ *The graphs have equal ratios, which may explain why they look like the same steepness.*

Classwork

Example 1 (2 minutes)

- Putting horizontal lines off to the side for a moment, there are two other types of non-vertical lines. Those that are *left-to-right inclining,* as in the graph of l_1, and those that are *left-to-right declining*, as in the graph of l_2, shown below.

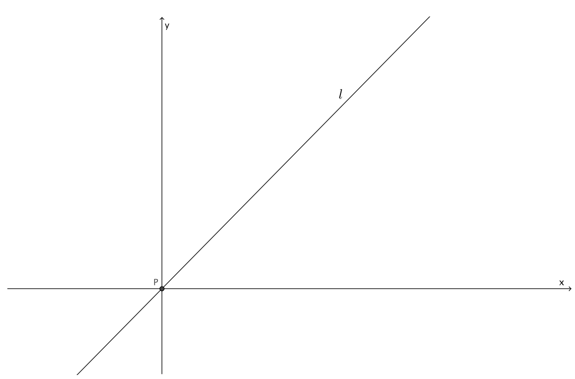




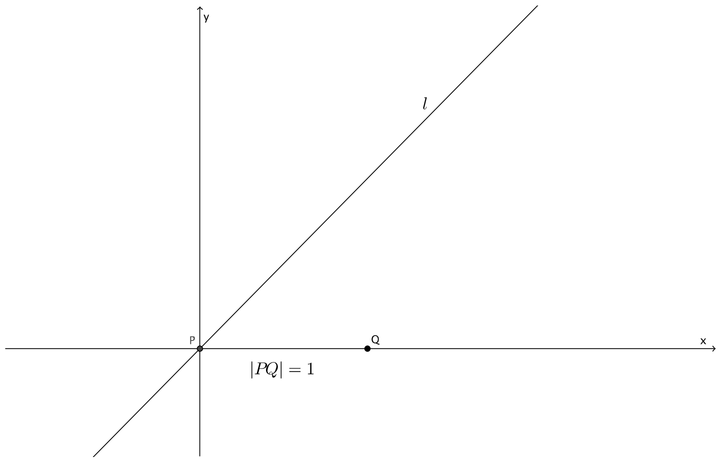
- We want to use a number to describe the amount of steepness or slant that each line has. The definition should be written in such a way that a horizontal line has a slope of 0, that is, no steepness and no slant.
- We begin by stating that lines that are *left-to-right inclining* are said to have a positive slope and lines that are *left-to-right declining* are said to have negative slope. We will discuss this more in a moment.

Example 2 (5 minutes)

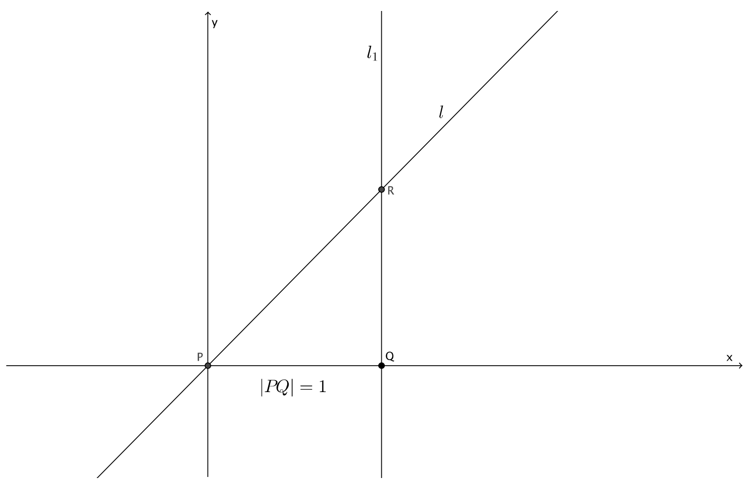
- Now let's look more closely at finding the number that will be the slope of a line. Suppose a non-vertical line l is given in the coordinate plane. We let P be the point on line l that goes through the origin.



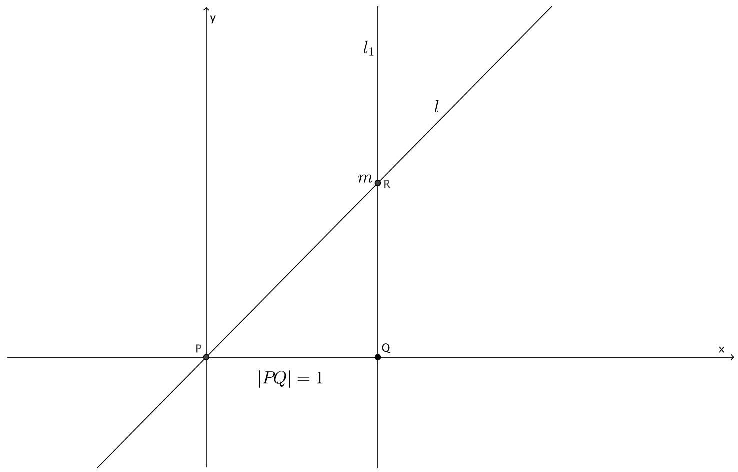
- Next, we locate a point Q, exactly one unit to the right of point P.



- Next, we draw a line, l_1, through point Q that is parallel to the y-axis. The point of intersection of line l and l_1 will be named point R.



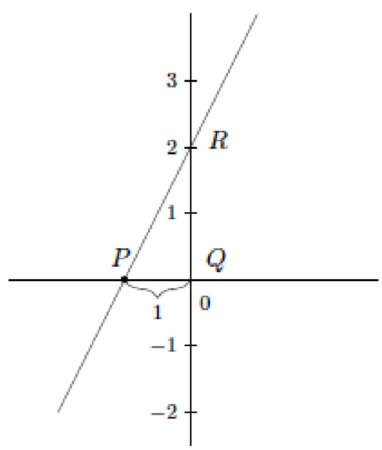
- Then the slope of line l is the number, m, associated with the y-coordinate of point R.



- The question remains, what is that number? Think of l_1 as a number line with point Q as zero, then however many units R is from Q is the number we are looking for, the slope m. Another way of thinking about this is through our basic rigid motion translation. We know that translation preserves lengths of segments. If we translate everything in the plane one unit to the left so that point Q maps onto the origin, then the segment QR will coincide with the y-axis and the y-coordinate of point R is the slope of the line.

Consider tracing the graph of the line and points P, Q, R onto a transparency. Demonstrate for students the translation along vector $\overrightarrow{QP}$ so that point Q is at the origin. Make clear that the translation moves point R to the y-axis, which is why the y-coordinate of point $R = (0, m)$ is the number that represents the slope of the line, m. This is how students would use the transparency, if needed, to complete the exercises in this lesson.

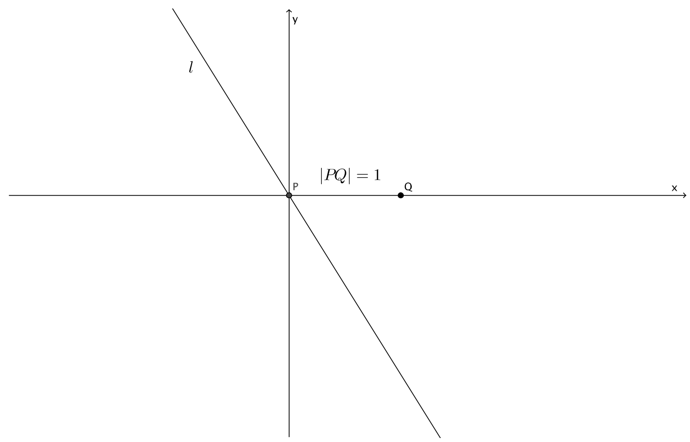
- Let's look at an example with real numbers. We have the same situation as just described. We have translated everything in the plane one unit to the left so that point Q maps onto the origin and the segment QR coincides with the y-axis. What is the slope of this line?



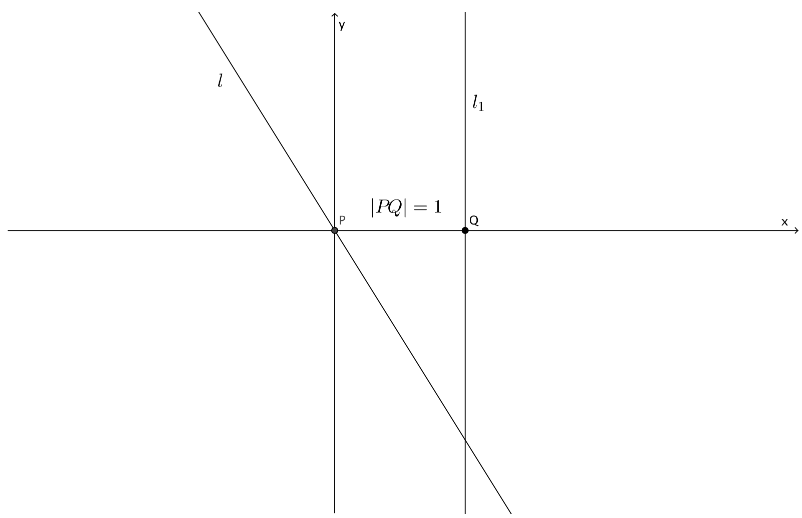
  - *The slope of the line is* 2*, or* $m = 2$.
- This explains why the slope of lines that are *left-to-right inclining* are positive. When we consider the number line, the point R is at positive 2; hence this line has a positive slope.

Example 3 (4 minutes)

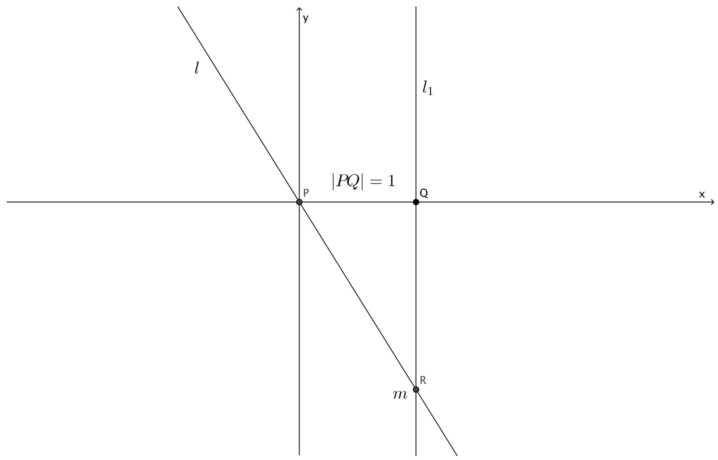
- Suppose a non-vertical line is given in the coordinate plane. As before, we mark a point P on the line and go one unit to the right of P and mark point Q.



- Then we draw a line, l_1, through point Q that is parallel to the y-axis.

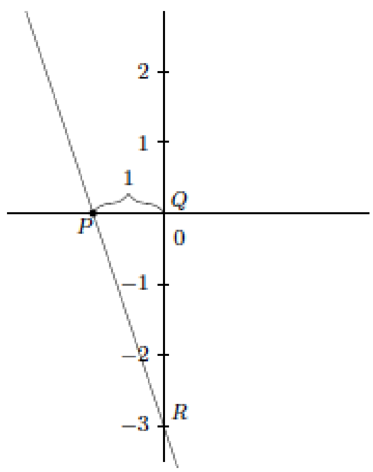


- We mark the intersection of lines l and l_1 as point R. Again, recall that we consider the line l_1 to be a vertical number line where point Q is at zero. Then the number associated with the y-coordinate of point R is the slope of the line.



Again, consider tracing the graph of the line and points P, Q, R onto a transparency. Demonstrate for students the translation along vector $\overrightarrow{QP}$ so that point Q is at the origin. Make clear that the translation moves point R to the y-axis, which is why the y-coordinate of point $R = (0, m)$ is the number that represents the slope of the line, m. This is how students would use the transparency, if needed, to complete the exercises in this lesson.

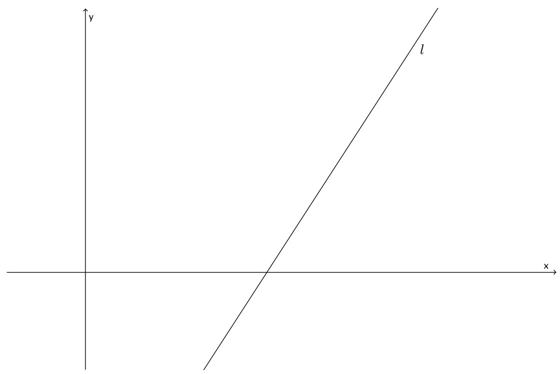
- Let's look at this example with real numbers. We have the same situation as just described. We have translated everything in the plane one unit to the left so that point Q maps onto the origin and the segment QR coincides with the y-axis. What is the slope of this line?



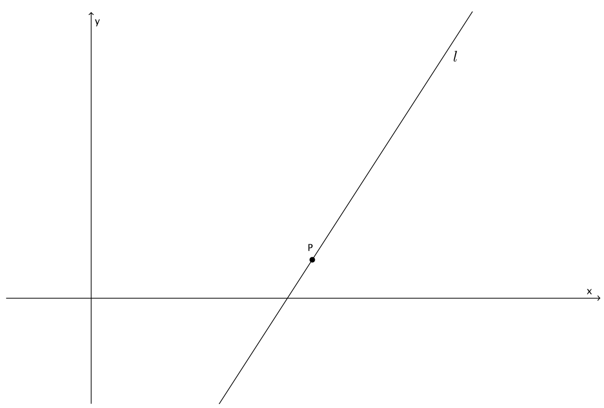
  - *The slope of the line is* -3*, or* $m = -3$.
- This explains why the slope of lines that are *left-to-right declining* are negative. When we consider the number line, the point R is at negative 3; hence this line has a negative slope.

Example 4 (4 minutes)

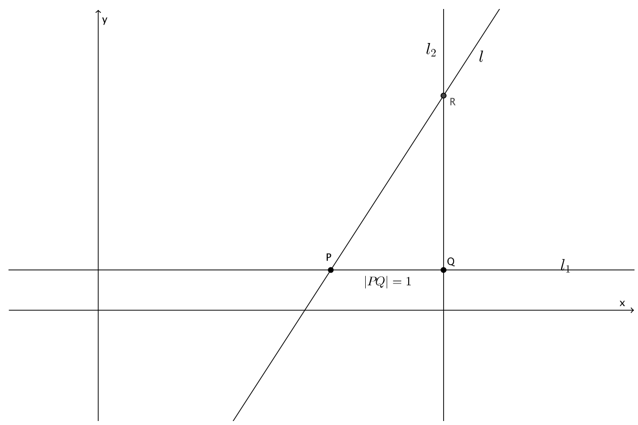
- Now we have a line l that does not go through the origin of the graph.



- Our process for finding slope changes only slightly. We will mark the point P at *any* location on the line l. Other than that, the work remains the same.



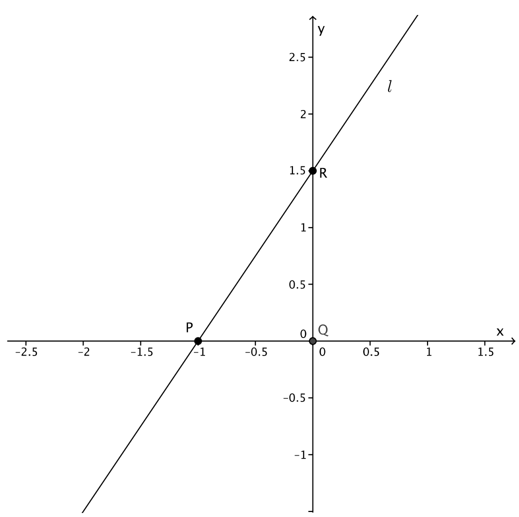
- We will go one unit to the right of P and mark point Q, then draw a line through Q parallel to the y-axis. We mark the intersection of lines l and l_1 point R. Again, recall that we consider the line l_1 to be a vertical number line where point Q is at zero. Then, the number associated with the y-coordinate of point R is the slope of the line.



- Just as before, we translate so that point Q maps onto the origin and the segment QR coincides with the y-axis.

Again, consider tracing the graph of the line and points P, Q, R onto a transparency. Demonstrate for students the translation so that point Q is at the origin (along a vector from Q to the origin). Make clear that the translation moves point R to the y-axis, which is why the y-coordinate of point $R = (0, m)$ is the number that represents the slope of the line, m. This is how students would use the transparency, if needed, to complete the exercises in this lesson.

- What is the slope of this line?



  - *The slope of the line is* 1.5*; that is,* $m = 1.5$.

- In general, we describe slope as an integer or a fraction. Given that we know the y-coordinate of R is 1.5, how can we express that number as a fraction?
  - *The slope of the line is* $\frac{3}{2}$*; that is,* $m = \frac{3}{2}$.

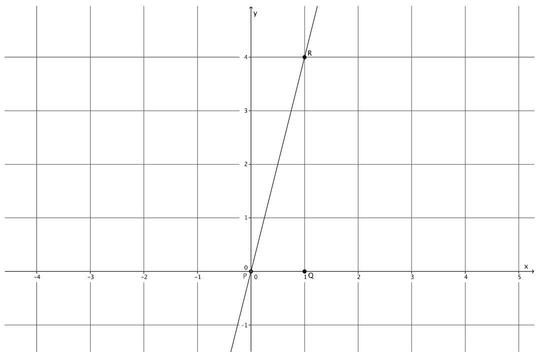
Exercises 1–6 (4 minutes)

Students complete Exercises 1–6 independently. The exercises are referenced in the discussion that follows.

Exercises 1–6

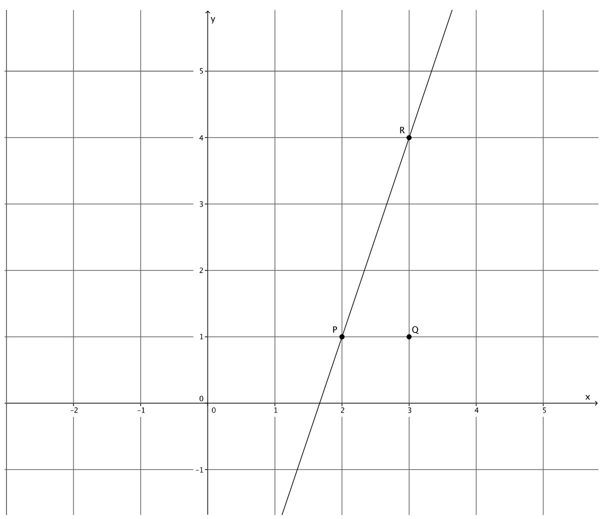
Use your transparency to find the slope of each line if needed.

1. **What is the slope of this non-vertical line?**



The slope of this line is 4, $m = 4$.

2. **What is the slope of this non-vertical line?**

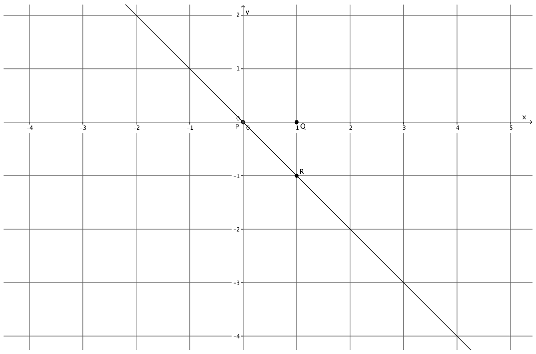


The slope of this line is 3, $m = 3$.

3. **Which of the lines in Exercises 1 and 2 is steeper? Compare the slopes of each of the lines. Is there a relationship between steepness and slope?**

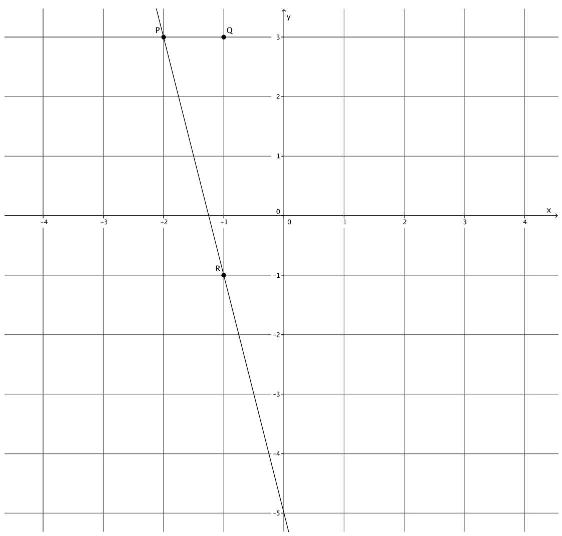
The graph in Exercise 1 seems steeper. The slopes are 4 *and* 3. *It seems like the greater the slope, the steeper the line.*

4. **What is the slope of this non-vertical line?**



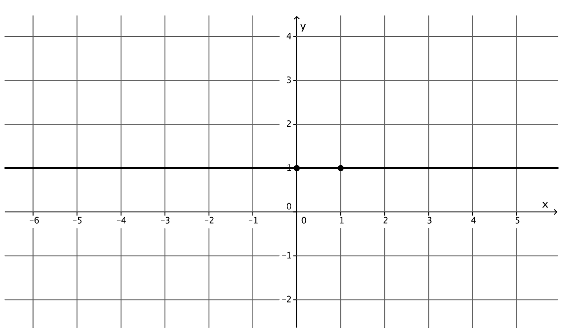
The slope of this line is -1, $m = -1$.

5. **What is the slope of this non-vertical line?**



The slope of this line is -4, $m = -4$.

6. **What is the slope of this non-vertical line?**



The slope of this line is 0, $m = 0$.

Discussion (6 minutes)

- When we began, we said that we wanted to define slope in a way so that a horizontal line would have a slope of 0 because a horizontal line has no steepness or slant. What did you notice in Exercise 6?
  - *The slope of the horizontal line was zero, like we wanted it to be.*
- In Exercise 3, you were asked to compare the steepness of the graphs of two lines and then compare their slopes. What did you notice?
  - *The steeper the line, the greater the number that describes the slope. For example, Exercise 1 had the steeper line and the greater slope.*
- Does this same relationship exist for lines with negative slopes? Look specifically at Exercises 4 and 5.

Provide students a minute or two to look back at Exercises 4 and 5. They should draw the conclusion that the absolute value of the slopes of lines that are *left-to-right* declining will determine which is steeper. Use the points below to bring this fact to light.

  - *A similar relationship exists. The line in Exercise 5 was steeper with a slope of* -4. *The line in Exercise 4 had a slope of* -1. *Based on our previous reasoning we'd say that because* $-1 > -4$, *that the line with a slope of* -1 *would have more steepness, but this is not the case.*

- We want to generalize the idea of steepness. When the slopes are positive we expect the line with greater steepness to have the greater slope. When the slopes are negative, it is actually the smaller number that has more steepness. Is there a way that we can compare the slopes so that our reasoning is consistent? That is, we want to say that a line is steeper than another when the number that describes the slope is larger than the other. How can we describe that mathematically, specifically when the slopes are negative?
  - *We can compare just the absolute value of the slopes. That way, we can say that the steeper line will be the slope with the greater absolute value.*

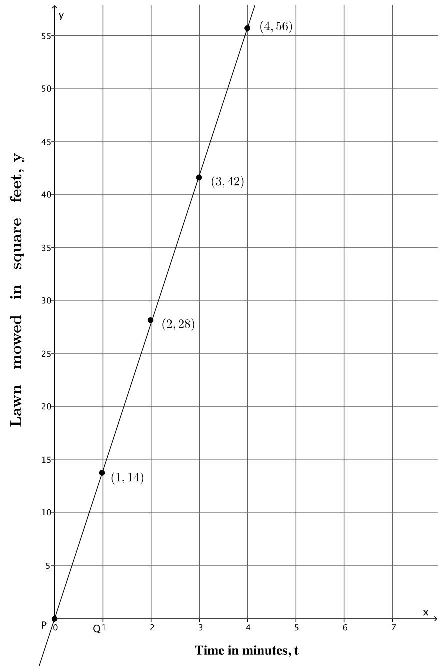
Example 5 (4 minutes)

- Let's take another look at one of the proportional relationships that we graphed in Lesson 11. Here is the problem and the work that we did:

  Pauline mows a lawn at a constant rate. Suppose she mowed a 35 square foot lawn in 2.5 minutes.

t (time in minutes)	Linear equation: $y = \frac{35}{2.5}t$	y (area in square feet)
0	$y = \frac{35}{2.5}(0)$	0
1	$y = \frac{35}{2.5}(1)$	$\frac{35}{2.5} = 14$
2	$y = \frac{35}{2.5}(2)$	$\frac{70}{2.5} = 28$
3	$y = \frac{35}{2.5}(3)$	$\frac{105}{2.5} = 42$
4	$y = \frac{35}{2.5}(4)$	$\frac{140}{2.5} = 56$

Now if we assume that the points we plot on the coordinate plane make a line and the origin of the graph is point P, then we have



- What is the slope of this line? Explain.
  - *One unit to the right of point P is the point Q. That makes the point $(1, 14)$ the location of point R. Therefore the slope of this line is 14, $m = 14$.*
- What is the unit rate of mowing the lawn?
  - *Pauline's unit rate of mowing a lawn is 14 square feet per 1 minute.*
- When we graph proportional relationships, the unit rate is interpreted as the slope of the graph of the line, which is why slope is referred to as the rate of change.

Closing (3 minutes)

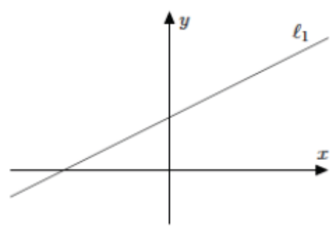
Summarize, or ask students to summarize, the main points from the lesson:

- We know that slope is a number that describes the steepness of a line.
- We know that lines that are left-to-right inclining have a positive slope and lines that are left-to-right declining have a negative slope.
- We can find the slope of a line by looking at the graph's unit rate.

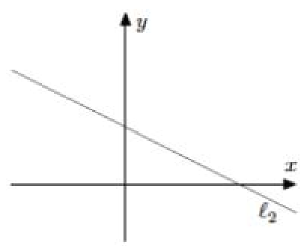
Lesson Summary

Slope is a number that describes the steepness of a line. Slope is represented by the symbol m.

Lines that are left-to-right inclining have a positive slope, as shown below.

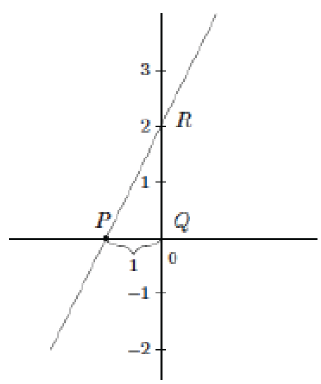


Lines that are left-to-right declining have a negative slope, as shown below.



Determine the slope of a line when the horizontal distance between points is fixed at 1 by translating point Q to the origin of the graph, and then identifying the y-coordinate of point R.

The slope of the line shown below is 2, or $m = 2$, because point R is at 2 on the y-axis.



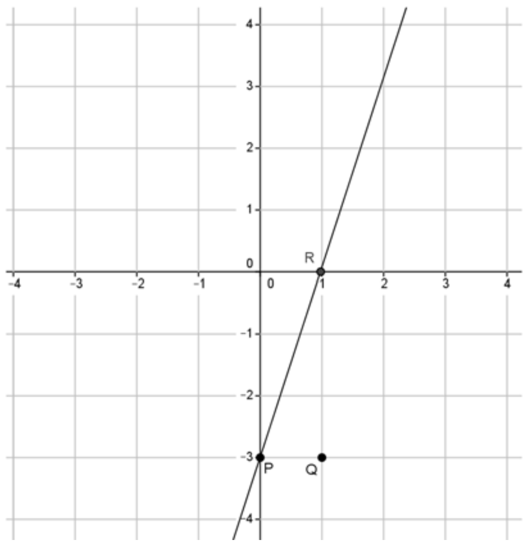
Exit Ticket (3 minutes)

Name ______________________________ Date ______________

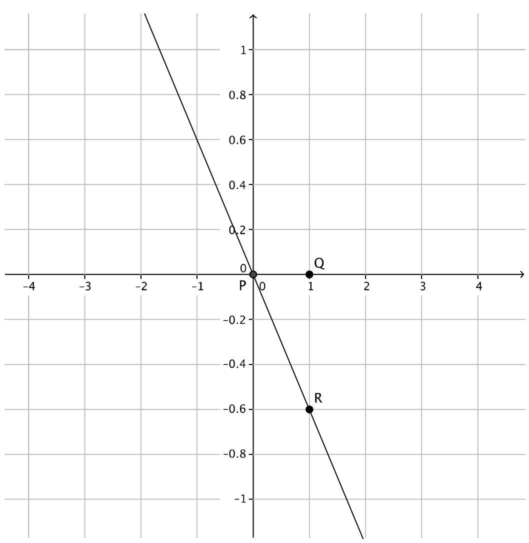
Lesson 15: The Slope of a Non-Vertical Line

Exit Ticket

1. What is the slope of this non-vertical line? Use your transparency if needed.

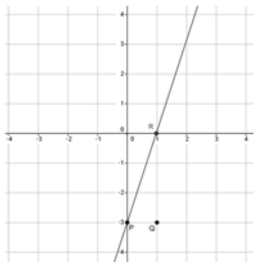


2. What is the slope of this non-vertical line? Use your transparency if needed.



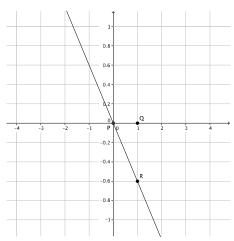
Exit Ticket Sample Solutions

1. **What is the slope of this non-vertical line? Use your transparency if needed.**



The slope of the line is 3, $m = 3$.

2. **What is the slope of this non-vertical line? Use your transparency if needed.**

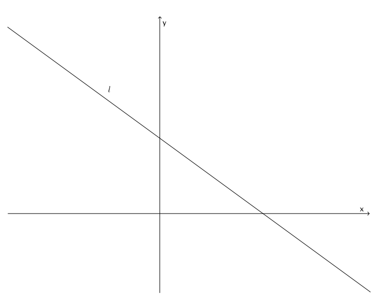


The slope of the line is -0.6 *which is equal to* $-\frac{3}{5}$, $m = -\frac{3}{5}$.

Problem Set Sample Solutions

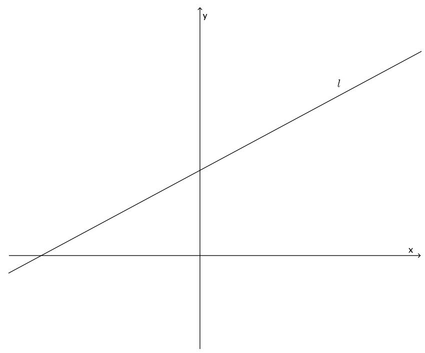
Students practice identifying graphs of lines as having positive or negative slope. Students interpret the unit rate of a graph as the slope of the graph. The following solutions indicate an understanding of the objectives of this lesson:

1. **Does the graph of the line shown below have a positive or negative slope? Explain.**



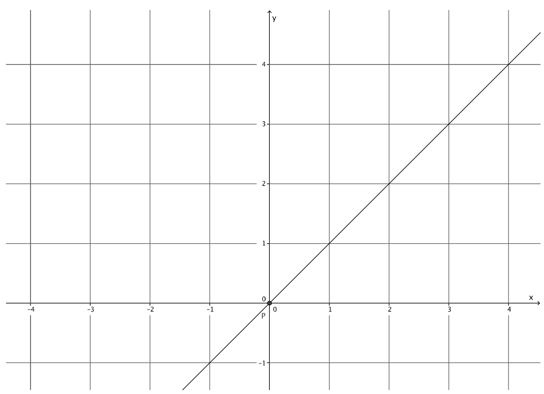
The graph of this line has a negative slope. First of all it is left-to-right declining, which is an indication of negative slope. Also, if we were to mark a point P *and a point* Q *one unit to the right of* P*, then draw a line parallel to the* y*-axis through* Q*, then the intersection of the two lines would be below* Q*, making the number that represents slope negative.*

2. **Does the graph of the line shown below have a positive or negative slope? Explain.**



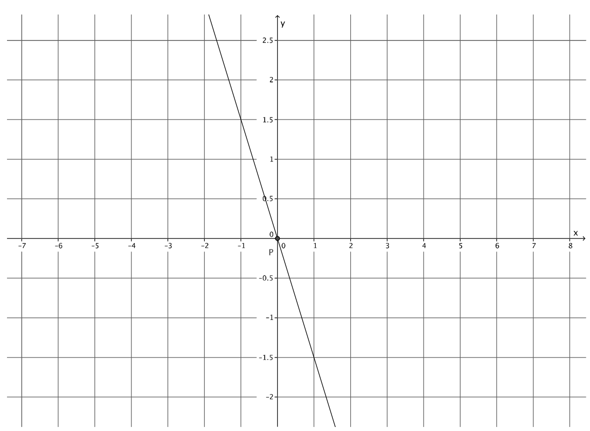
The graph of this line has a positive slope. First of all it is left-to-right inclining, which is an indication of positive slope. Also, if we were to mark a point P and a point Q one unit to the right of P, then draw a line parallel to the y-axis through Q, then the intersection of the two lines would be above Q, making the number that represents slope positive.

3. **What is the slope of this non-vertical line? Use your transparency if needed.**



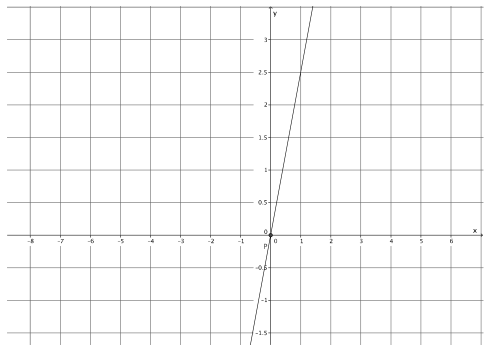
The slope of this line is 1, $m = 1$.

4. **What is the slope of this non-vertical line? Use your transparency if needed.**



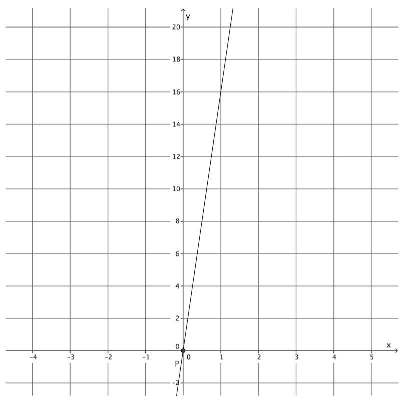
The slope of this line is $-\frac{3}{2}$, $m = -\frac{3}{2}$.

5. **What is the slope of this non-vertical line? Use your transparency if needed.**



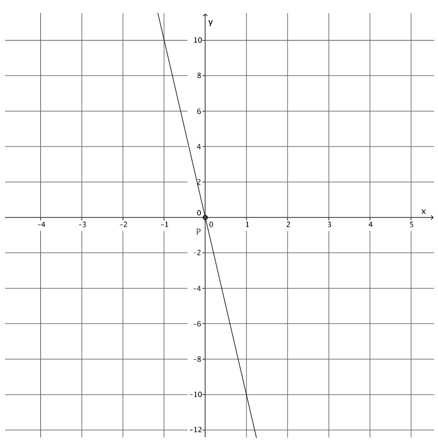
The slope of this line is $\frac{5}{2}$, $m = \frac{5}{2}$.

6. **What is the slope of this non-vertical line? Use your transparency if needed.**



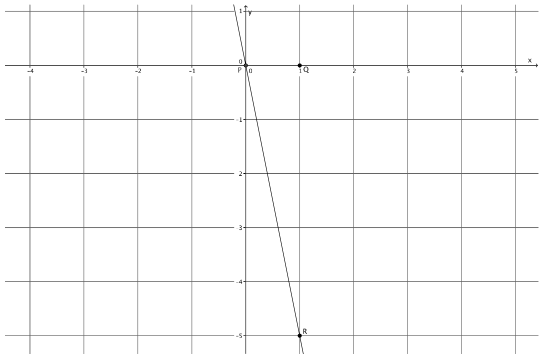
The slope of this line is 16, $m = 16$.

7. **What is the slope of this non-vertical line? Use your transparency if needed.**



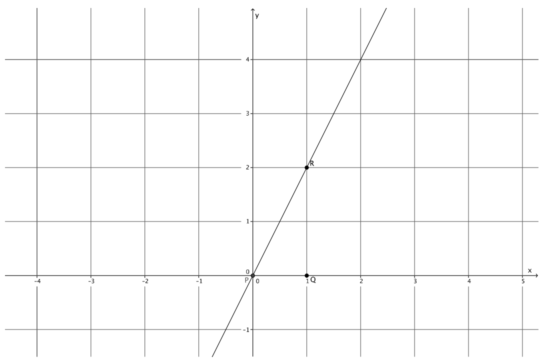
The slope of this line is -10, $m = -10$.

8. **What is the slope of this non-vertical line? Use your transparency if needed.**



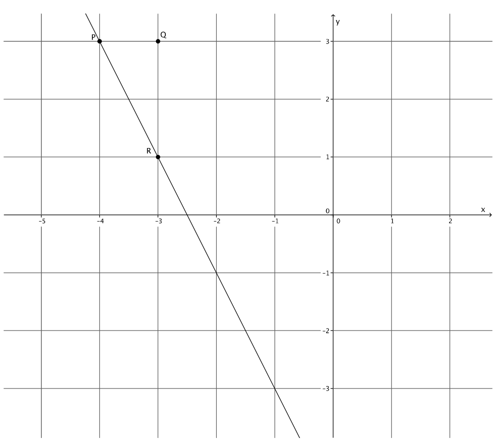
The slope of this line is -5, $m = -5$.

9. **What is the slope of this non-vertical line? Use your transparency if needed.**



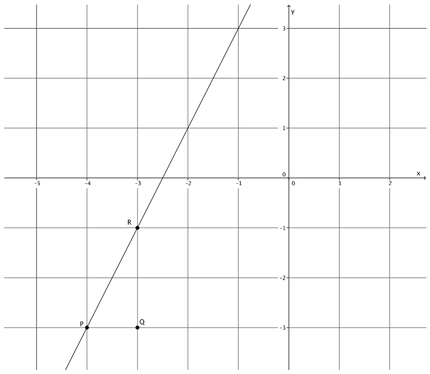
The slope of this line is 2, $m = 2$.

10. **What is the slope of this non-vertical line? Use your transparency if needed.**



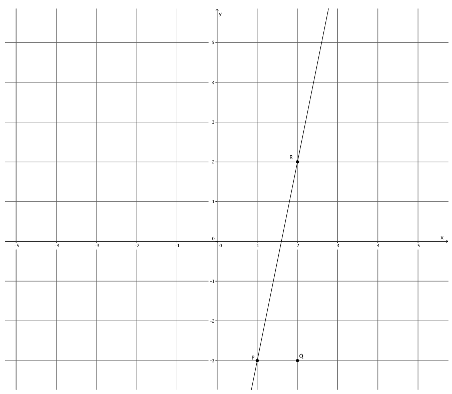
The slope of this line is -2, $m = -2$.

11. What is the slope of this non-vertical line? Use your transparency if needed.



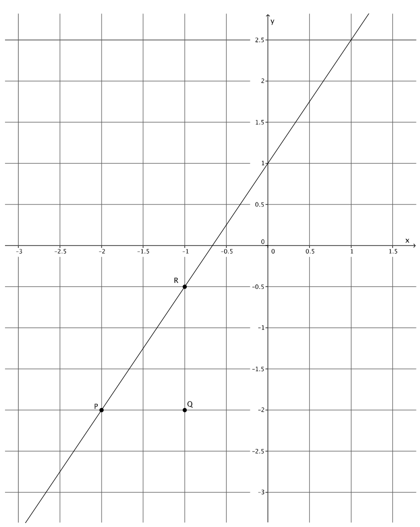
The slope of this line is 2, $m = 2$.

12. What is the slope of this non-vertical line? Use your transparency if needed.



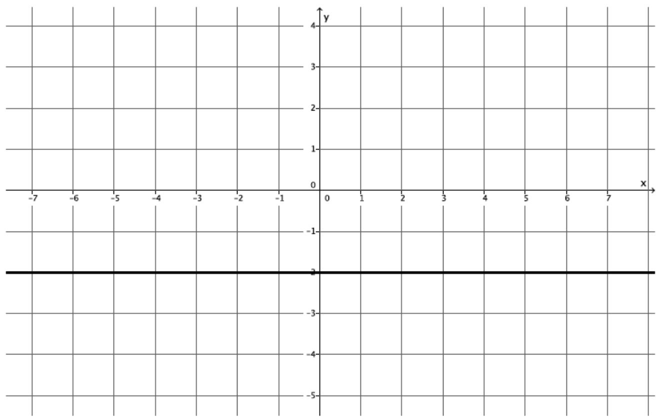
The slope of this line is 5, $m = 5$.

13. What is the slope of this non-vertical line? Use your transparency if needed.



The slope of this line is $\frac{3}{2}$, $m = \frac{3}{2}$.

14. **What is the slope of this non-vertical line? Use your transparency if needed.**

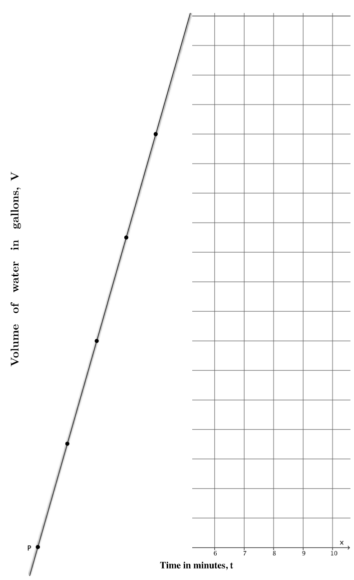


The slope of this line is 0, $m = 0$.

In Lesson 11, you did the work below involving constant rate problems. Use the table and the graphs provided to answer the questions that follow.

15. **Suppose the volume of water that comes out in three minutes is 10.5 gallons.**

t (time in minutes)	Linear equation: $V = \frac{10.5}{3}t$	V (in gallons)
0	$V = \frac{10.5}{3}(0)$	0
1	$V = \frac{10.5}{3}(1)$	$\frac{10.5}{3} = 3.5$
2	$V = \frac{10.5}{3}(2)$	$\frac{21}{3} = 7$
3	$V = \frac{10.5}{3}(3)$	$\frac{31.5}{3} = 10.5$
4	$V = \frac{10.5}{3}(4)$	$\frac{42}{3} = 14$



a. **How many gallons of water flow out of the faucet per minute, that is, what is the unit rate of water flow?**

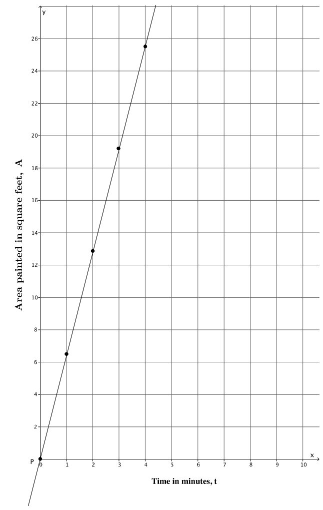
The unit rate of water flow is 3.5 gallons per minute.

b. **Assume that the graph of the situation is a line, as shown in the graph. What is the slope of the line?**

The slope of the line is 3.5, $m = 3.5$.

16. Emily paints at a constant rate. She can paint 32 square feet in five minutes.

t (time in minutes)	Linear equation: $A = \frac{32}{5}t$	A (area painted in square feet)
0	$A = \frac{32}{5}(0)$	0
1	$A = \frac{32}{5}(1)$	$\frac{32}{5} = 6.4$
2	$A = \frac{32}{5}(2)$	$\frac{64}{5} = 12.8$
3	$A = \frac{32}{5}(3)$	$\frac{96}{5} = 19.2$
4	$A = \frac{32}{5}(4)$	$\frac{128}{5} = 25.6$



a. How many square feet can Emily paint in one minute; that is, what is her unit rate of painting?

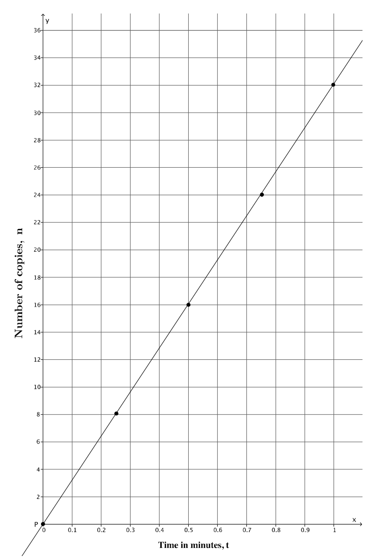
The unit rate at which Emily paints is 6.4 square feet per minute.

b. Assume that the graph of the situation is a line, as shown in the graph. What is the slope of the line?

The slope of the line is 6.4, $m = 6.4$.

17. A copy machine makes copies at a constant rate. The machine can make 80 copies in $2\frac{1}{2}$ minutes.

t (time in minutes)	Linear equation: $n = 32t$	n (number of copies)
0	$n = 32(0)$	0
0.25	$n = 32(0.25)$	8
0.5	$n = 32(0.5)$	16
0.75	$n = 32(0.75)$	24
1	$n = 32(1)$	32



a. How many copies can the machine make each minute; that is, what is the unit rate of the copy machine?

The unit rate of the copy machine is 32 copies per minute.

b. Assume that the graph of the situation is a line, as shown in the graph. What is the slope of the line?

The slope of the line is 32, $m = 32$.