Lesson 23: The Defining Equation of a Line

Student Outcomes

- Students know that two equations in the form of $ax + by = c$ and $a'x + b'y = c'$ graph as the same line when $\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}$ and at least one of a or b is nonzero.
- Students know that the graph of a linear equation $ax + by = c$, where a, b, and c are constants and at least one of a or b is nonzero, is the line defined by the equation $ax + by = c$.

Lesson Notes

Following the Exploratory Challenge is a discussion that presents a theorem about the defining equation of a line (page 351) and then a proof of the theorem. The proof of the theorem is optional. The discussion can end with the theorem and in place of the proof, students can complete Exercises 4–8 beginning on page 353. Whether you choose to discuss the proof or have students complete Exercises 4–8, it is important that students understand that two equations that are written differently can be the same and their graph will be the same line. This reasoning becomes important when you consider systems of linear equations. In order to make sense of "infinitely many solutions" to a system of linear equations, students must know that equations that might appear different can have the same graph and represent the same line. Further, students should be able to recognize when two equations define the same line without having to graph each equation, which is the goal of this lesson. Students need graph paper to complete the Exploratory Challenge.

Classwork

Exploratory Challenge/Exercises 1–3 (20 minutes)

Students need graph paper to complete the Exercises in the Exploratory Challenge. Students complete Exercises 1–3 in pairs or small groups.

Exploratory Challenge/Exercises 1–3

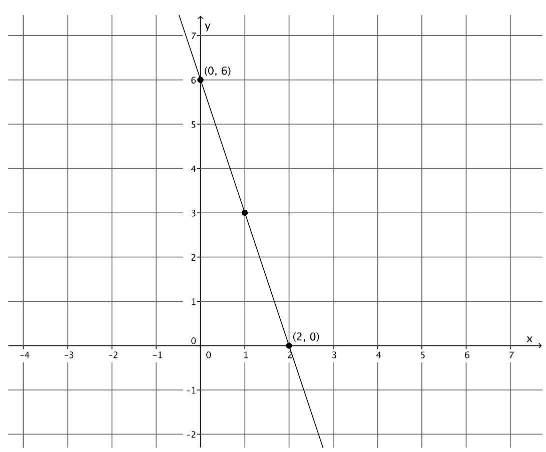
1. **Graph the equation $9x + 3y = 18$ using intercepts. Then answer parts (a)–(f) that follow.**

$$9(0) + 3y = 18$$
$$3y = 18$$
$$y = 6$$

The y-intercept is $(0, 6)$.

$$9x + 3(0) = 18$$
$$9x = 18$$
$$x = 2$$

The x-intercept is $(2, 0)$.



a. Graph the equation $y = -3x + 6$ on the same coordinate plane.

b. What do you notice about the graphs of $9x + 3y = 18$ and $y = -3x + 6$? Why do you think this is so?

The graphs of the equations produce the same line. Both equations go through the same two points so they are the same line.

c. Rewrite $y = -3x + 6$ in standard form.

$$y = -3x + 6$$
$$3x + y = 6$$

d. Identify the constants, a, b, c of the equation in standard form from part (c).

$a = 3$, $b = 1$, *and* $c = 6$.

e. Identify the constants of the equation $9x + 3y = 18$. Note them as a', b', and c'.

$a' = 9$, $b' = 3$, *and* $c' = 18$

f. What do you notice about $\frac{a'}{a}$, $\frac{b'}{b}$, and $\frac{c'}{c}$?

$$\frac{a'}{a} = \frac{9}{3} = 3, \quad \frac{b'}{b} = \frac{3}{1} = 3, \quad and \quad \frac{c'}{c} = \frac{18}{6} = 3$$

Each fraction is equal to the number 3.

2. Graph the equation $y = \frac{1}{2}x + 3$ using the y-intercept and the slope. Then answer parts (a)–(f) that follow.

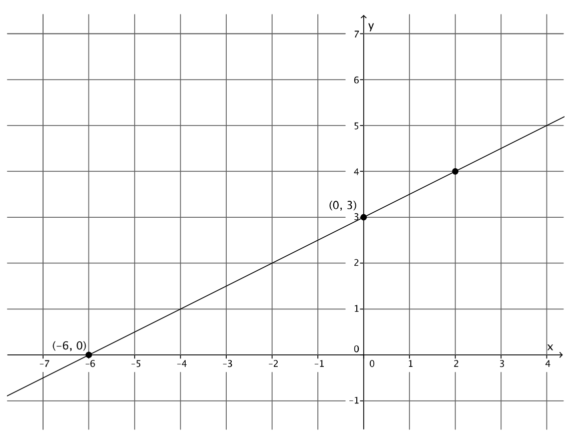
a. Graph the equation $4x - 8y = -24$ using intercepts on the same coordinate plane.

$$4(0) - 8y = -24$$
$$-8y = -24$$
$$y = 3$$

The y-intercept is $(0, 3)$.

$$4x - 8(0) = -24$$
$$4x = -24$$
$$x = -6$$

The x-intercept is $(-6, 0)$.



b. What do you notice about the graphs of $y = \frac{1}{2}x + 3$ and $4x - 8y = -24$? Why do you think this is so?

The graphs of the equations produce the same line. Both equations go through the same two points, so they are the same line.

c. Rewrite $y = \frac{1}{2}x + 3$ in standard form.

$$y = \frac{1}{2}x + 3$$
$$\left(y = \frac{1}{2}x + 3\right)2$$
$$2y = x + 6$$
$$-x + 2y = 6$$
$$-1(-x + 2y = 6)$$
$$x - 2y = -6$$

d. Identify the constants, a, b, c of the equation in standard form from part (c).

$a = 1$, $b = -2$, and $c = -6$.

e. Identify the constants of the equation $4x - 8y = -24$. Note them as a', b', and c'.

$a' = 4$, $b' = -8$, and $c' = -24$

f. What do you notice about $\frac{a'}{a}$, $\frac{b'}{b}$, and $\frac{c'}{c}$?

$\frac{a'}{a} = \frac{4}{1} = 4$, $\frac{b'}{b} = \frac{-8}{-2} = 4$, *and* $\frac{c'}{c} = \frac{-24}{-6} = 4$

Each fraction is equal to the number 4.

3. The equations $y = \frac{2}{3}x - 4$ and $6x - 9y = 36$ graph as the same line.

a. Rewrite $y = \frac{2}{3}x - 4$ in standard form.

$$y = \frac{2}{3}x - 4$$
$$\left(y = \frac{2}{3}x - 4\right)3$$
$$3y = 2x - 12$$
$$-2x + 3y = -12$$
$$-1(-2x + 3y = -12)$$
$$2x - 3y = 12$$

b. Identify the constants, a, b, c, of the equation in standard form from part (a).

$a = 2$, $b = -3$, and $c = 12$.

c. Identify the constants of the equation $6x - 9y = 36$. Note them as a', b', and c'.

$a' = 6$, $b' = -9$, and $c' = 36$

d. What do you notice about $\frac{a'}{a}$, $\frac{b'}{b}$, and $\frac{c'}{c}$?

$\frac{a'}{a} = \frac{6}{2} = 3$, $\frac{b'}{b} = \frac{-9}{-3} = 3$, *and* $\frac{c'}{c} = \frac{36}{12} = 3$

Each fraction is equal to the number 3.

e. **You should have noticed that each fraction was equal to the same constant. Multiply that constant by the standard form of the equation from part (a). What do you notice?**

$$2x - 3y = 12$$
$$3(2x - 3y = 12)$$
$$6x - 9y = 36$$

After multiplying the equation from part (a) by 3, I noticed that it is the exact same equation that was given.

Discussion (15 minutes)

Following the statement of the theorem is an optional proof of the theorem. Below the proof are Exercises 4–8 (beginning on page 353) that can be completed instead of the proof.

MP.8

- What did you notice about the equations you graphed in each of the Exercises 1–3?
  - *In each case, the equations graphed as the same line.*
- What you observed in Exercises 1–3 can be summarized in the following theorem:

Theorem. *Suppose a, b, c, a', b', and c' are constants,* where at least one of a or b is nonzero, and one of a' or b' is nonzero .

(1) *If there is a non-zero number s, so that $a' = sa$, $b' = sb$, and $c' = sc$, then the graphs of the equations $ax + by = c$ and $a'x + b'y = c'$ are the same line.*

(2) *If the graphs of the equations $ax + by = c$ and $a'x + b'y = c'$ are the same line, then there exists a nonzero number s so that $a' = sa$, $b' = sb$, and $c' = sc$.*

The optional part of the discussion begins here.

- We want to show that (1) is true. We need to show that the equations $ax + by = c$ and $a'x + b'y = c'$ are the same. What information are we given in (1) that will be useful in showing that the two equations are the same?
  - *We are given that $a' = sa$, $b' = sb$, and $c' = sc$. We can use substitution in the equation $a'x + b'y = c'$ since we know what a', b', and c' are equal to.*

- Then by substitution, we have

$$a'x + b'y = c'$$
$$sax + sby = sc$$

By the distributive property

$$s(ax + by) = sc$$

Divide both sides of the equation by s

$$ax + by = c$$

Which is what we wanted $a'x + b'y = c'$ to be equal to. Therefore, we have proved (1).

- To prove (2), we will assume that $a, b \neq 0$, that is, we are not dealing with horizontal or vertical lines. Proving (2) will require us to rewrite the given equations, $ax + by = c$ and $a'x + b'y = c'$, in slope-intercept form. Rewrite the equations.
  - $ax + by = c$

$$by = -ax + c$$
$$\frac{b}{b}y = -\frac{a}{b}x + \frac{c}{b}$$
$$y = -\frac{a}{b}x + \frac{c}{b}$$

  - $a'x + b'y = c'$

$$b'y = -a'x + c'$$
$$\frac{b'}{b'}y = -\frac{a'}{b'}x + \frac{c'}{b'}$$
$$y = -\frac{a'}{b'}x + \frac{c'}{b'}$$

- We will refer to $y = -\frac{a}{b}x + \frac{c}{b}$ as (A) and $y = -\frac{a'}{b'}x + \frac{c'}{b'}$ as (B).

- What are the slopes of (A) and (B)?
  - *The slope of (A) is $-\frac{a}{b}$ and the slope of (B) is $-\frac{a'}{b'}$.*

- Since we know that the two lines are the same, we know the slopes must be the same.

$$-\frac{a}{b} = -\frac{a'}{b'}$$

and when we multiply both sides of the equation by (-1), we have

$$\frac{a}{b} = \frac{a'}{b'}$$

By using the multiplication property of equality we can rewrite $\frac{a}{b} = \frac{a'}{b'}$ as

$$\frac{a'}{a} = \frac{b'}{b}$$

Notice that this proportion is equivalent to the original form.

Scaffolding:

Students may need to see the intermediate steps in the rewriting of $\frac{a}{b} = \frac{a'}{b'}$ as $\frac{a'}{a} = \frac{b'}{b}$.

- What are the y-intercepts of $y = -\frac{a}{b}x + \frac{c}{b}$ (A) and $y = -\frac{a'}{b'}x + \frac{c'}{b'}$ (B)?
  - *The y-intercept of (A) is $\frac{c}{b}$, and the y-intercept of (B) is $\frac{c'}{b'}$.*

- Because we know that the two lines are the same, the y-intercepts will be the same.

$$\frac{c}{b} = \frac{c'}{b'}$$

We can rewrite the proportion.

$$\frac{c'}{c} = \frac{b'}{b}$$

Using the transitive property and the fact that $\frac{a'}{a} = \frac{b'}{b}$ and $\frac{c'}{c} = \frac{b'}{b}$, we can make the following statement:

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}$$

Therefore, all three ratios are equal to the same number. Let this number be s.

$$\frac{a'}{a} = s, \frac{b'}{b} = s, \frac{c'}{c} = s$$

We can rewrite this to state that $a' = sa$, $b' = sb$, and $c' = cs$. Therefore, (2) is proved.

- When two equations are the same, i.e., they graph as the same line, we say that any one of those equations is the *defining equation* of the line.

Exercises 4–8 (15 minutes)

Students complete Exercises 4–8 independently or in pairs. Consider having students share the equations they write for each exercise while others in the class verify which equations graph as the same line.

Exercises 4–8

4. Write three equations that would graph as the same line as the equation $3x + 2y = 7$.

Answers will vary. Verify that students have multiplied a, b, and c by the same constant when they write the new equation.

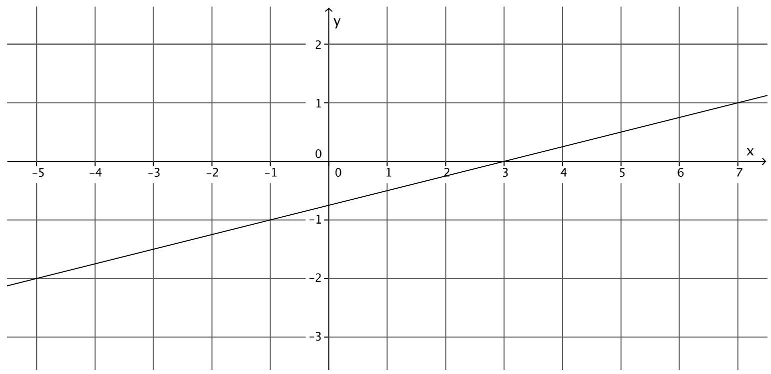
5. Write three equations that would graph as the same line as the equation $x - 9y = \frac{3}{4}$.

Answers will vary. Verify that students have multiplied a, b, and c by the same constant when they write the new equation.

6. Write three equations that would graph as the same line as the equation $-9x + 5y = -4$.

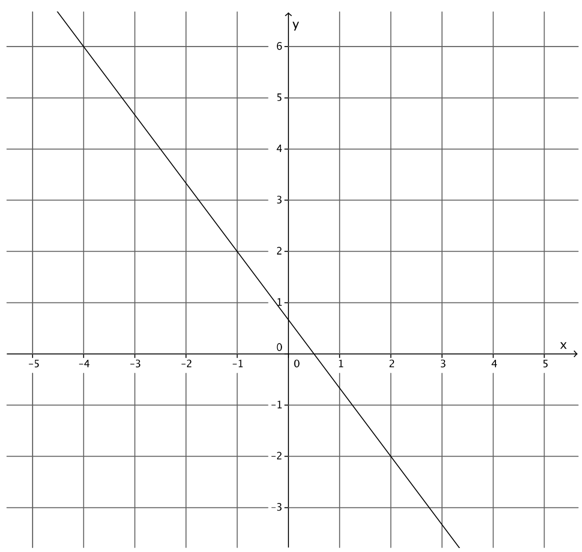
Answers will vary. Verify that students have multiplied a, b, and c by the same constant when they write the new equation.

7. **Write at least two equations in the form $ax + by = c$ that would graph as the line shown below.**



Answers will vary. Verify that students have the equation $-x + 4y = -3$ in some form.

8. **Write at least two equations in the form $ax + by = c$ that would graph as the line shown below.**



Answers will vary. Verify that students have the equation $4x + 3y = 2$ in some form.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that when two equations graph as the same line it is because they are the same equation in different forms.
- We know that even if the equations that graph as the same line look different (i.e., different constants or different forms) that any one of those equations can be referred to as the defining equation of the line.

Lesson Summary

Two equations that graph as the same line are said to define the same line. Two equations that define the same line are the same equation, just in different forms. The equations may look different (different constants, different coefficients, or different forms).

When two equations are written in standard form, $ax + by = c$ and $a'x + b'y = c'$, they define the same line when $\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}$ is true.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 23: The Defining Equation of a Line

Exit Ticket

1. Do the equations $-16x + 12y = 33$ and $-4x + 3y = 8$ graph as the same line? Why or why not?

2. Given the equation $3x - y = 11$, write another equation that will graph as the same line. Explain why.

Exit Ticket Sample Solutions

1. **Do the equations $-16x + 12y = 33$ and $-4x + 3y = 8$ graph as the same line? Why or why not?**

   No, in the first equation $a = -16$, $b = 12$, and $c = 33$, and in the second equation $a' = -4$, $b' = 3$, and $c' = 8$. Then,

   $$\frac{a'}{a} = \frac{-4}{-16} = \frac{1}{4}, \quad \frac{b'}{b} = \frac{3}{12} = \frac{1}{4},$$

   $$\text{but } \frac{c'}{c} = \frac{8}{33} = \frac{8}{33}.$$

   Since each fraction does not equal the same number, then they do not graph to the same line.

2. **Given the equation $3x - y = 11$, write another equation that will graph as the same line. Explain why.**

   Answers will vary. Verify that the student has written an equation that defines the same line by showing that the fractions $\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = s$, where s is some constant.

Problem Set Sample Solutions

Students practice identifying pairs of equations as the defining equation of a line or two distinct lines.

1. **Do the equations $x + y = -2$ and $3x + 3y = -6$ define the same line? Explain.**

   Yes, these equations define the same line. When you compare the constants from each equation you get:

   $$\frac{a'}{a} = \frac{3}{1} = 3, \quad \frac{b'}{b} = \frac{3}{1} = 3, \quad \text{and} \quad \frac{c'}{c} = \frac{-6}{-2} = 3$$

   When I multiply the first equation by 3, I get the second equation:

   $$(x + y = -2)3$$
   $$3x + 3y = -6$$

   Therefore, these equations define the same line.

2. **Do the equations $y = -\frac{5}{4}x + 2$ and $10x + 8y = 16$ define the same line? Explain.**

   Yes, these equations define the same line. When you rewrite the first equation in standard form you get:

   $$y = -\frac{5}{4}x + 2$$
   $$\left(y = -\frac{5}{4}x + 2\right)4$$
   $$4y = -5x + 8$$
   $$5x + 4y = 8$$

   When you compare the constants from each equation you get:

   $$\frac{a'}{a} = \frac{10}{5} = 2, \quad \frac{b'}{b} = \frac{8}{4} = 2, \quad \text{and} \quad \frac{c'}{c} = \frac{16}{8} = 2$$

   When I multiply the first equation by 2, I get the second equation:

   $$(5x + 4y = 8)2$$
   $$10x + 8y = 16$$

   Therefore, these equations define the same line.

3. Write an equation that would define the same line as $7x - 2y = 5$.

Answers will vary. Verify that the student has written an equation that defines the same line by showing that the fractions $\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = s$, where s is some constant.

4. Challenge: Show that if the two lines given by $ax + by = c$ and $a'x + b'y = c'$ are the same when $b = 0$ (vertical lines), then there exists a non-zero number s, so that $a' = sa$, $b' = sb$, and $c' = sc$.

When $b = 0$, then the equations are $ax = c$ and $a'x = c'$. We can rewrite the equations as $x = \frac{c}{a}$ and $x = \frac{c'}{a'}$. Because the equations graph as the same line, then we know that

$$\frac{c}{a} = \frac{c'}{a'}$$

and we can rewrite those fractions as

$$\frac{a'}{a} = \frac{c'}{c}$$

These fractions are equal to the same number. Let that number be s. Then $\frac{a'}{a} = s$ and $\frac{c'}{c} = s$, and therefore, $a' = sa$ and $c' = sc$, as desired.

5. Challenge: Show that if the two lines given by $ax + by = c$ and $a'x + b'y = c'$ are the same when $a = 0$ (horizontal lines), then there exists a non-zero number s, so that $a' = sa$, $b' = sb$, and $c' = sc$.

When $a = 0$, then the equations are $by = c$ and $b'y = c'$. We can rewrite the equations as $y = \frac{c}{b}$ and $y = \frac{c'}{b'}$. Because the equations graph as the same line, then we know that their slopes are the same.

$$\frac{c}{b} = \frac{c'}{b'}$$

We can rewrite the proportion.

$$\frac{b'}{b} = \frac{c'}{c}$$

These fractions are equal to the same number. Let that number be s. Then $\frac{b'}{b} = s$ and $\frac{c'}{c} = s$. Therefore, $b' = sb$ and $c' = sc$.