Lesson 3: Linear Functions and Proportionality

Student Outcomes

- Students relate constant speed and proportional relationships to linear functions using information from a table.
- Students know that distance traveled is a function of the time spent traveling and that the total cost of an item is a function of how many items are purchased.

Classwork

Example 1 (7 minutes)

Example 1

In the last lesson we looked at several tables of values that represented the inputs and outputs of functions. For example:

Bags of Candy (x)	1	2	3	4	5	6	7	8
Cost (y)	$\$1.25$	$\$2.50$	$\$3.75$	$\$5$	$\$6.25$	$\$7.50$	$\$8.75$	$\$10$

- What do you think a *linear* function is?
  - *A linear function is likely a function with a linear relationship. Specifically, the rate of change is constant and the graph is a line.*
- In the last lesson, we looked at several tables of values that represented the inputs and outputs of functions. For example:

Bags of Candy (x)	1	2	3	4	5	6	7	8
Cost (y)	$1.25	$2.50	$3.75	$5	$6.25	$7.50	$8.75	$10

- Do you think this is a *linear* function? Justify your answer.
  - *Yes, this is a linear function because there is a constant rate of change, as shown below:*

$$\frac{\$10}{8\ bags\ of\ candy} = \$1.25\ per\ each\ bag\ of\ candy$$

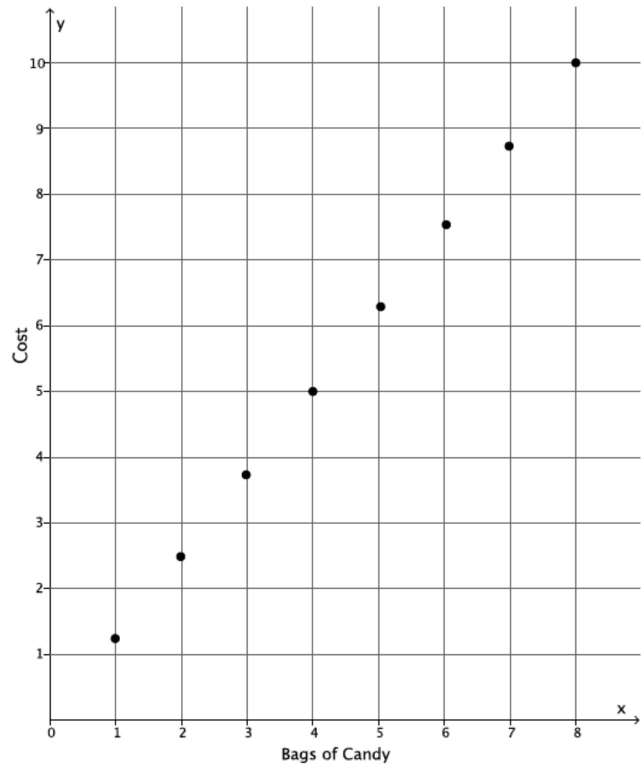
$$\frac{\$5}{4\ bags\ of\ candy} = \$1.25\ per\ each\ bag\ of\ candy$$

$$\frac{\$2.50}{2\ bags\ of\ candy} = \$1.25\ per\ each\ bag\ of\ candy$$

Scaffolding:

In addition to explanations about functions, it may be useful for students to have a series of structured experiences with real-world objects and data to reinforce their understanding of a function. An example is experimenting with different numbers of "batches" of a given recipe; students can observe the effect of the number of batches on quantities of various ingredients.

- *The total cost is increasing at a rate of* $\$1.25$ *with each bag of candy. Further proof comes from the graph of the data shown below.*



- A linear function is a function with a rule so that the output is equal to m multiplied by the input plus b, where m and b are fixed constants. If y is the output and x is the input, then a linear function is represented by the rule $y = mx + b$. That is, when the rule that describes the function is in the form of $y = mx + b$, then the function is a linear function. Notice that this is not any different from a linear equation in two variables. What rule or equation describes this function?
  - *The rule that represents the function is then* $y = 1.25x$.
- Notice that the constant m is 1.25, which is the cost of one bag of candy, and the constant b is zero. Also notice that the constant m was found by calculating the unit rate for a bag of candy. What we know of linear functions so far is no different than what we learned about linear equations—the unit rate of a proportional relationship is the rate of change.
- No matter what value of x is chosen, as long as x is a non-negative integer, the rule $y = 1.25x$ represents the cost function of a bag of candy. Moreover, *the total cost of candy is a function of the number of bags purchased*.
- Why do we have to note that x is a non-negative integer for this function?
  - *Since* x *represents the number of bags of candy, it does not make sense that there would be a negative number of bags. For that reason,* x *as a positive integer means the function allows us to find the cost of zero or more bags of candy.*
- Would you say that the table represents all possible inputs and outputs? Explain.
  - *No, it does not represent all possible inputs and outputs. I'm sure someone can purchase more than* 8 *bags of candy, and inputs greater than* 8 *are not represented by this table.*

- As a matter of precision, we say that "this function has the above table of values" instead of "the table above represents a function" because not all values of the function can be represented by the table. The rule, or formula, that describes the function can represent all of the possible values of a function. For example, using the rule we could determine the cost for 9 bags of candy. However, this statement should not lead you to believe that a table cannot entirely represent a function. In this context, if there were a limit on the number of bags that could be purchased, i.e., 8 bags, then the table above would represent the function completely.

Example 2 (4 minutes)

Example 2

Walter walks 8 miles in two hours. What is his average speed?

- Consider the following rate problem: Walter walks 8 miles in two hours. What is his average speed?
  - *Walter's average speed of walking 8 miles is $\frac{8}{2} = 4$, or 4 miles per hour.*
- If we assume constant speed, then we can determine the distance Walter walks over any time period using the equation $y = 4x$, where y is the distance walked in x hours. Walter's rate of walking is constant; therefore, no matter what x is, we can say that the distance Walter walks is a linear function given by the equation $y = 4x$. Again, notice that the constant m of $y = mx + b$ is 4, which represents the unit rate of walking for Walter.

Scaffolding:

As the language becomes more abstract, it can be useful to use visuals and even pantomime situations related to speed, rate, etc.

- In the last example, the total cost of candy was a function of the number of bags purchased. Describe the function in this example.
  - *The distance that Walter travels is a function of the number of hours he spends walking.*
- What limitations do we need to put on x?
  - *The limitation that we should put on x is that $x \geq 0$. Since x represents the time Walter walks, then it makes sense that he would walk for a positive amount of time or no time at all.*
- Since x is positive, then we know that the distance y will also be positive.

Example 3 (4 minutes)

Example 3

Veronica runs at a constant speed. The distance she runs is a function of the time she spends running. The function has the table of values shown below.

Time in Minutes (x)	8	16	24	32
Distance Ran in Miles (y)	1	2	3	4

- Veronica runs at a constant speed. The distance she runs is a function of the time she spends running. The function has the table of values shown below.

Time in Minutes (x)	8	16	24	32
Distance Ran in Miles (y)	1	2	3	4

- Since Veronica runs at a constant speed, we know that her average speed over any time interval will be the same. Therefore, Veronica's distance function is a linear function. Write the equation that describes her distance function.
  - *The function that represents Veronica's distance is described by the equation $y = \frac{1}{8}x$, where y is the distance in miles Veronica runs in x minutes and $x, y \geq 0$.*
- Describe the function in terms of distance and time.
  - *The distance that Veronica runs is a function of the number of minutes she spends running.*

Example 4 (5 minutes)

Example 4

Water flows from a faucet at a constant rate. That is, the volume of water that flows out of the faucet is the same over any given time interval. If 7 gallons of water flow from the faucet every 2 minutes, determine the rule that describes the volume function of the faucet.

The rate of water flow is $\frac{7}{2}$, 3.5 gallons per minute. Then the rule that describes the volume function of the faucet is $y = 3.5x$, where y is the volume of water that flows from the faucet and x is the number of minutes the faucet is on.

- Assume that the faucet is filling a bathtub that can hold 50 gallons of water. How long will it take the faucet to fill the tub?
  - *Since we want the total volume to be 50 gallons, then*

$$50 = 3.5x$$
$$\frac{50}{3.5} = x$$
$$14.2857\ldots = x$$
$$14 \approx x$$

It will take about 14 minutes to fill a tub that has a volume of 50 gallons.

Now assume that you are filling the same tub, a tub with a volume of 50 gallons, with the same faucet, a faucet where the rate of water flow is 3.5 gallons per minute. This time, however, the tub already has 8 gallons in it. Will it still take 14 minutes to fill the tub? Explain.

No, it will take less time because there is already some water in the tub.

- How can we reflect the water that is already in the tub with our volume of water flow as a function of time for the faucet?
  - *If y is the volume of water that flows from the faucet and x is the number of minutes the faucet is on, then $y = 3.5x + 8$.*
- How long will it take the faucet to fill the tub if the tub already has 8 gallons in it?
  - *Since we still want the total volume of the tub to be 50 gallons, then*

$$50 = 3.5x + 8$$
$$42 = 3.5x$$
$$12 = x$$

It will take 12 minutes for the faucet to fill a 50 gallon tub when 8 gallons are already in it.

- Generate a table of values for this function:

Time in Minutes (x)	0	3	6	9	12
Total Volume in Tub in Gallons (y)	8	18.5	29	39.5	50

Example 5 (7 minutes)

Example 5

Water flows from a faucet at a constant rate. Assume that 6 gallons of water are already in a tub by the time we notice the faucet is on. This information is recorded as 0 minutes and 6 gallons of water in the table below. The other values show how many gallons of water are in the tub at the given number of minutes.

Time in Minutes (x)	0	3	5	9
Total Volume in Tub in Gallons (y)	6	9.6	12	16.8

- After 3 minutes pass, there are 9.6 gallons in the tub. How much water flowed from the faucet in those 3 minutes? Explain.
  - *Since there were already 6 gallons in the tub, after 3 minutes an additional 3.6 gallons filled the tub.*
- Use this information to determine the rate of water flow.
  - *In 3 minutes, 3.6 gallons were added to the tub, then $\frac{3.6}{3} = 1.2$, and the faucet fills the tub at a rate of 1.2 gallons per minute.*
- Verify that the rate of water flow is correct using the other values in the table.
  - *Sample student work:*

    $5(1.2) = 6$ *and since 6 gallons were already in the tub, the total volume in the tub is 12 gallons.*

    $9(1.2) = 10.8$ *and since 6 gallons were already in the tub, the total volume in the tub is 16.8 gallons.*
- Write the volume of water flow as a function of time that represents the rate of water flow from the faucet.
  - *The volume function that represents the rate of water flow from the faucet is $y = 1.2x$, where y is the volume of water that flows from the faucet and x is the number of minutes the faucet is on.*

- Write the rule or equation that describes the volume of water flow as a function of time for filling the tub, including the 6 gallons that are already in the tub to begin with.
  - *Since the tub already has* 6 *gallons it in, then the rule is* $y = 1.2x + 6$.
- How many minutes was the faucet on before we noticed it? Explain.
  - *Since* 6 *gallons were in the tub by the time we noticed the faucet was on, we need to determine how many minutes it takes for* 6 *gallons to flow from the faucet:*

$$6 = 1.2x$$
$$5 = x$$

The faucet was on for 5 *minutes before we noticed it.*

Exercises 1–3 (10 minutes)

Students complete Exercises 1–3 independently or in pairs.

Exercises 1–3

1. A linear function has the table of values below. The information in the table shows the function of time in minutes with respect to mowing an area of lawn in square feet.

Number of Minutes (x)	5	20	30	50
Area Mowed in Square Feet (y)	36	144	216	360

a. Explain why this is a linear function.

Sample responses:

Linear functions have a constant rate of change. When we compare the rates at each interval of time, they will be equal to the same constant.

When the data is graphed on the coordinate plane, it appears to make a line.

b. Describe the function in terms of area mowed and time.

The total area mowed is a function of the number of minutes spent mowing.

c. What is the rate of mowing a lawn in 5 minutes?

$$\frac{36}{5} = 7.2$$

The rate is 7.2 square feet per minute.

d. What is the rate of mowing a lawn in 20 minutes?

$$\frac{144}{20} = 7.2$$

The rate is 7.2 square feet per minute.

e. **What is the rate for mowing a lawn in 30 minutes?**

$$\frac{216}{30} = 7.2$$

The rate is 7.2 square feet per minute.

f. **What is the rate for mowing a lawn in in 50 minutes?**

$$\frac{360}{50} = 7.2$$

The rate is 7.2 square feet per minute.

g. **Write the rule that represents the linear function that describes the area in square feet mowed, y, in x minutes.**

$$y = 7.2x$$

h. **Describe the limitations of x and y.**

Both x and y must be positive numbers. The symbol x represents time spent mowing, which means it should be positive. Similarly, y represents the area mowed, which should also be positive.

i. **What number does the function assign to 24? That is, what area of lawn can be mowed in 24 minutes?**

$$y = 7.2(24)$$
$$y = 172.8$$

In 24 minutes, an area of 172.8 square feet can be mowed.

j. **How many minutes would it take to mow an area of 400 square feet?**

$$400 = 7.2x$$
$$\frac{400}{7.2} = x$$
$$55.555\ldots = x$$
$$56 \approx x$$

It would take about 56 minutes to mow an area of 400 square feet.

2. **A linear function has the table of values below. The information in the table shows the volume of water that flows from a hose in gallons as a function of time in minutes.**

Time in Minutes (x)	10	25	50	70
Total Volume of Water in Gallons (y)	44	110	220	308

a. **Describe the function in terms of volume and time.**

The total volume of water that flows from a hose is a function of the number of minutes the hose is left on.

b. Write the rule that represents the linear function that describes the volume of water in gallons, y, in x minutes.

$$y = \frac{44}{10}x$$
$$y = 4.4x$$

c. What number does the function assign to 250? That is, how many gallons of water flow from the hose in 250 minutes?

$$y = 4.4(250)$$
$$y = 1,100$$

In 250 minutes, $1,100$ gallons of water flow from the hose.

d. The average pool has about $17,300$ gallons of water. The pool has already been filled $\frac{1}{4}$ of its volume. Write the rule that describes the volume of water flow as a function of time for filling the pool using the hose, including the number of gallons that are already in the pool.

$$\frac{1}{4}(17,300) = 4,325$$
$$y = 4.4x + 4,325$$

e. Approximately how much time, in hours, will it take to finish filling the pool?

$$17,300 = 4.4x + 4,325$$
$$12,975 = 4.4x$$
$$\frac{12,975}{4.4} = x$$
$$2,948.8636\ldots = x$$
$$2,949 \approx x$$
$$\frac{2,949}{60} = 49.15$$

It will take about 49 hours to fill the pool with the hose.

3. Recall that a linear function can be described by a rule in the form of $y = mx + b$, where m and b are constants. A particular linear function has the table of values below.

Input (x)	0	4	10	11	15	20	23
Output (y)	4	24	54	59	79	104	119

a. What is the equation that describes the function?

$$y = 5x + 4$$

b. Complete the table using the rule.

Closing (4 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that a linear function can be described by a rule in the form of $y = mx + b$, where m and b are constants.
- We know that constant rates and proportional relationships can be described by a linear function.
- We know that the distance traveled is a function of the time spent traveling, that the volume of water flow from a faucet is a function of the time the faucet is on, etc.

Lesson Summary

Functions can be described by a rule in the form of $y = mx + b$, where m and b are constants.

Constant rates and proportional relationships can be described by a function, specifically a linear function where the rule is a linear equation.

Functions are described in terms of their inputs and outputs. For example, if the inputs are related to time and the output are distances traveled at given time intervals then we say that the distance traveled is a function of the time spent traveling.

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 3: Linear Functions and Proportionality

Exit Ticket

1. A linear function has the table of values below. The information in the tables shows the number of pages a student can read in a certain book as a function of time in minutes. Assume a constant rate.

Time in Minutes (x)	2	6	11	20
Total Number of Pages Read in a Certain Book (y)	7	21	38.5	70

   a. Write the rule or equation that represents the linear function that describes the total number of pages read, y, in x minutes.

   b. How many pages can be read in 45 minutes?

   c. This certain book has 396 pages. The student has already read $\frac{3}{8}$ of the pages. Write the equation that describes the number of pages read as a function of time for reading this book, including the number pages that have already been read.

   d. Approximately how much time, in minutes, will it take to finish reading the book?

Exit Ticket Sample Solutions

1. A linear function has the table of values below. The information in the tables shows the number of pages a student can read in a certain book as a function of time in minutes. Assume a constant rate.

Time in Minutes (x)	2	6	11	20
Total Number of Pages Read in a Certain Book (y)	7	21	38.5	70

a. Write the rule or equation that represents the linear function that describes the total number of pages read, y, in x minutes.

$$y = \frac{7}{2}x$$
$$y = 3.5x$$

b. How many pages can be read in 45 minutes?

$$y = 3.5(45)$$
$$y = 157.5$$

In 45 minutes, the student can read 157.5 pages.

c. This certain book has 396 pages. The student has already read $\frac{3}{8}$ of the pages. Write the equation that describes the number of pages read as a function of time for reading this book, including the number pages that have already been read.

$$\frac{3}{8}(396) = 148.5$$
$$y = 3.5x + 148.5$$

d. Approximately how much time, in minutes, will it take to finish reading the book?

$$398 = 3.5x + 148.5$$
$$249.5 = 3.5x$$
$$\frac{249.5}{3.5} = x$$
$$71.285714\ldots = x$$
$$71 \approx x$$

It will take about 71 minutes to finish reading the book.

Problem Set Sample Solutions

1. A food bank distributes cans of vegetables every Saturday. They keep track of the cans in the following manner in the table. A linear function can be used to represent the data. The information in the table shows the function of time in weeks to the number of cans of vegetables distributed by the food bank.

Number of Weeks (x)	1	12	20	45
Number of Cans of Vegetables Distributed (y)	180	$2,160$	$3,600$	$8,100$

a. Describe the function in terms of cans distributed and time.

The total number of cans handed out is a function of the number of weeks that pass.

b. Write the equation or rule that represents the linear function that describes the number of cans handed out, y, in x weeks.

$$y = \frac{180}{1}x$$
$$y = 180x$$

c. Assume that the food bank wants to distribute $20,000$ cans of vegetables. How long will it take them to meet that goal?

$$20,000 = 180x$$
$$\frac{20,000}{180} = x$$
$$111.1111\ldots = x$$
$$111 \approx x$$

It will take about 111 weeks to distribute $20,000$ cans of vegetables or about 2 years.

d. Assume that the food bank has already handed out $35,000$ cans of vegetables and continues to hand out cans at the same rate each week. Write a linear function that accounts for the number of cans already handed out.

$$y = 180x + 35,000$$

e. Using your function in part (c), determine how long in years it will take the food bank to hand out $80,000$ cans of vegetables.

$$80,000 = 180x + 35,000$$
$$45,000 = 180x$$
$$\frac{45,000}{180} = x$$
$$250 = x$$
$$\frac{250}{52} = \text{number of years}$$
$$4.8076\ldots = \text{number of years}$$
$$4.8 \approx \text{number of years}$$

It will take about 4.8 years to distribute $80,000$ cans of vegetables.

2. A linear function has the table of values below. The information in the table shows the function of time in hours to the distance an airplane travels in miles. Assume constant speed.

Number of Hour Traveled (x)	2.5	4	4.2
Distance in Miles (y)	$1,062.5$	1700	$1,785$

a. Describe the function in terms of distance and time.

The total distance traveled is a function of the number of hours spent flying.

b. Write the rule that represents the linear function that describes the distance traveled in miles, y, in x hours.

$$y = \frac{1,062.5}{2.5}x$$
$$y = 425x$$

c. Assume that the airplane is making a trip from New York to Los Angeles which is approximately $2,475$ miles. How long will it take the airplane to get to Los Angeles?

$$2,475 = 425x$$
$$\frac{2,475}{425} = x$$
$$5.82352\ldots = x$$
$$5.8 \approx x$$

It will take about 5.8 hours for the airplane to fly $2,475$ miles.

d. The airplane flies for 8 hours. How many miles will it be able to travel in that time interval?

$$y = 425(8)$$
$$y = 3,400$$

The airplane would travel $3,400$ miles in 8 hours.

3. A linear function has the table of values below. The information in the table shows the function of time in hours to the distance a car travels in miles.

Number of Hours Traveled (x)	3.5	3.75	4	4.25
Distance in Miles (y)	203	217.5	232	246.5

a. Describe the function in terms of area distance and time.

The total distance traveled is a function of the number of hours spent traveling.

b. Write the rule that represents the linear function that describes the distance traveled in miles, y, in x hours.

$$y = \frac{203}{3.5}x$$
$$y = 58x$$

c. Assume that the person driving the car is going on a road trip that is 500 miles from their starting point. How long will it take them to get to their destination?

$$500 = 58x$$
$$\frac{500}{58} = x$$
$$8.6206\ldots = x$$
$$8.6 \approx x$$

It will take about 8.6 hours to travel 500 miles.

d. Assume that a second car is going on the road trip from the same starting point and traveling at the same rate. However, this car has already driven 210 miles. Write the rule that represents the linear function that accounts for the miles already driven by this car.

$$y = 58x + 210$$

e. How long will it take the second car to drive the remainder of the trip?

$$500 = 58x + 210$$
$$290 = 58x$$
$$\frac{290}{58} = x$$
$$5 = x$$

It will take 5 hours to drive the remaining 290 miles of the road trip.

4. A particular linear function has the table of values below.

Input (x)	2	3	8	11	15	20	23
Output (y)	7	10	25	34	46	61	70

a. What is the equation that describes the function?

$$y = 3x + 1$$

b. Complete the table using the rule.

5. A particular linear function has the table of values below.

Input (x)	0	5	8	13	15	18	21
Output (y)	6	11	14	19	21	24	27

a. What is the rule that describes the function?

$$y = x + 6$$

b. Complete the table using the rule.