Lesson 8: The Long Division Algorithm

Student Outcomes

- Students know that the long division algorithm is the basic skill to get division-with-remainder and the decimal expansion of a number in general.
- Students know why digits repeat in terms of the algorithm.
- Students know that every rational number has a decimal expansion that repeats eventually.

Lesson Notes

In this lesson we move towards being able to define an irrational number by formalizing the definition of a rational number.

Classwork

Example 1 (5 minutes)

Example 1

Show that the decimal expansion of $\frac{26}{4}$ is 6.5.

Scaffolding:

There is no single long division algorithm. The algorithm commonly taught and used in the U.S. is rarely used elsewhere. Students may come with earlier experiences with other division algorithms that make more sense to them. Consider using formative assessment to determine how different students approach long division.

Use the Example with students so they have a model to complete Exercises 1–5.

MP.3

- Show that the decimal expansion of $\frac{26}{4}$ is 6.5.
  - *Students will most likely use the long division algorithm.*
- Division is really just another form of multiplication. Here is a demonstration of that fact: Let's consider the fraction $\frac{26}{4}$ in terms of multiplication. We want to know the greatest number of groups of 4 that are in 26. How many are there?
  - *There are 6 groups of 4 in 26.*
- Is there anything leftover, a remainder?
  - *Yes, there are 2 leftover.*
- Symbolically, we can express the number 26 as:

$$26 = 6 \times 4 + 2$$

MP.3

- With respect to the fraction $\frac{26}{4}$ we can represent the division as

$$\frac{26}{4} = \frac{6 \times 4 + 2}{4}$$
$$\frac{26}{4} = \frac{6 \times 4}{4} + \frac{2}{4}$$
$$\frac{26}{4} = 6 + \frac{2}{4}$$
$$\frac{26}{4} = 6\frac{2}{4} = 6\frac{1}{2}$$

- The fraction $\frac{26}{4}$ is equal to the finite decimal 6.5. When the fraction is not equal to a finite decimal, then we need to use the long division algorithm to determine the decimal expansion of the number.

Exploratory Challenge

Exercises 1–5 (15 minutes)

Students complete Exercises 1–5 independently or in pairs. The discussion that follows is related to the concepts in the Exercises.

Exercises 1–5

1. Use long division to determine the decimal expansion of $\frac{142}{2}$.

$$2\overline{)142.0} = 71.0$$

a. Fill in the blanks to show another way to determine the decimal expansion of $\frac{142}{2}$.

$$142 = \underline{71} \times 2 + \underline{0}$$
$$\frac{142}{2} = \frac{\underline{71} \times 2 + \underline{0}}{2}$$
$$\frac{142}{2} = \frac{\underline{71} \times 2}{2} + \frac{\underline{0}}{2}$$
$$\frac{142}{2} = \underline{71} + \frac{\underline{0}}{2}$$
$$\frac{142}{2} = \underline{71.0}$$

b. Does the number $\frac{142}{2}$ have a finite or infinite decimal expansion? Explain how you know.

The decimal expansion of $\frac{142}{2}$ is 71.0 and is finite because the denominator of the fraction, 2, can be expressed as a product of 2's.

2. Use long division to determine the decimal expansion of $\frac{142}{4}$.

$$4\overline{)142.0} = 35.5$$

a. Fill in the blanks to show another way to determine the decimal expansion of $\frac{142}{4}$.

$$142 = \underline{35} \times 4 + \underline{2}$$

$$\frac{142}{4} = \frac{\underline{35} \times 4 + \underline{2}}{4}$$

$$\frac{142}{4} = \frac{\underline{35} \times 4}{4} + \frac{\underline{2}}{4}$$

$$\frac{142}{4} = \underline{35} + \frac{\underline{2}}{4}$$

$$\frac{142}{4} = \underline{35\frac{2}{4} = 35.5}$$

b. Does the number $\frac{142}{4}$ have a finite or infinite decimal expansion? Explain how you know.

The decimal expansion of $\frac{142}{4}$ is 35.5 and is finite because the denominator of the fraction, 4, can be expressed as a product of 2's.

3. Use long division to determine the decimal expansion of $\frac{142}{6}$.

```
    23.666
6)142.000
  12
   22
   18
    40
    36
     40
     36
      40
      36
       4
```

a. Fill in the blanks to show another way to determine the decimal expansion of $\frac{142}{6}$.

$$142 = \underline{23} \times 6 + \underline{4}$$

$$\frac{142}{6} = \frac{\underline{23} \times 6 + \underline{4}}{6}$$

$$\frac{142}{6} = \frac{\underline{23} \times 6}{6} + \frac{\underline{4}}{6}$$

$$\frac{142}{6} = \underline{23} + \frac{\underline{4}}{6}$$

$$\frac{142}{6} = \underline{23\frac{4}{6} = 23.666\ldots}$$

b. Does the number $\frac{142}{6}$ have a finite or infinite decimal expansion? Explain how you know.

The decimal expansion of $\frac{142}{6}$ is $23.666\ldots$ and is infinite because the denominator of the fraction, 6, cannot be expressed as a product of 2's and/or 5's.

4. Use long division to determine the decimal expansion of $\frac{142}{11}$.

$$\begin{array}{r} 12.90909 \\ 11\overline{)142.00000} \\ \underline{11} \\ 32 \\ \underline{22} \\ 100 \\ \underline{99} \\ 10 \\ \underline{00} \\ 100 \\ \underline{99} \\ 10 \\ \underline{00} \\ 100 \\ \underline{99} \\ 10 \end{array}$$

a. Fill in the blanks to show another way to determine the decimal expansion of $\frac{142}{11}$.

$$142 = \underline{\ 12\ } \times 11 + \underline{\ 10\ }$$

$$\frac{142}{11} = \frac{\underline{\ 12\ } \times 11 + \underline{\ 10\ }}{11}$$

$$\frac{142}{11} = \frac{\underline{\ 12\ } \times 11}{11} + \frac{\underline{\ 10\ }}{11}$$

$$\frac{142}{11} = \underline{\ 12\ } + \frac{\underline{\ 10\ }}{11}$$

$$\frac{142}{11} = \underline{12\frac{10}{11} = 12.90909\ldots}$$

b. Does the number $\frac{142}{11}$ have a finite or infinite decimal expansion? Explain how you know.

The decimal expansion of $\frac{142}{11}$ is $12.90909\ldots$ and is infinite because the denominator of the fraction, 6, cannot be expressed as a product of 2's and/or 5's.

5. Which fractions produced an infinite decimal expansion? Why do you think that is?

The fractions that required the long division algorithm to determine the decimal expansion were $\frac{142}{6}$ and $\frac{142}{11}$. The fact that these numbers had an infinite decimal expansion is due to the fact that the divisor was not a product of 2's and/or 5's compared to the first two fractions where the divisor was a product of 2's and/or 5's. In general, the decimal expansion of a number will be finite when the divisor, i.e., the denominator of the fraction, can be expressed as a product of 2's and/or 5's. Similarly, the decimal expansion will be infinite when the divisor cannot be expressed as a product of 2's and/or 5's.

Discussion (10 minutes)

- What is the decimal expansion of $\frac{142}{2}$?

If students respond "71", ask them what decimal digits they could include without changing the value of the number.

  - *The fraction* $\frac{142}{2}$ *is equal to the decimal* $71.00000 \ldots$
- Did you need to use the long division algorithm to determine your answer? Why or why not?
  - *No, the long division algorithm was not necessary because there was a whole number of* 2*'s in* 142*.*
- What is the decimal expansion of $\frac{142}{4}$?
  - *The fraction* $\frac{142}{4}$ *is equal to the decimal* 35.5*.*
- What decimal digits could we include to the right of the 0.5 without changing the value?
  - *We could write the decimal as* $35.500000 \ldots$
- Did you need to use the long division algorithm to determine your answer? Why or why not?
  - *No, the long division algorithm was not necessary because* $\frac{142}{4} = 35 + \frac{2}{4}$ *and* $\frac{2}{4}$ *is a finite decimal. We could use what we learned in the last lesson to write* $\frac{2}{4}$ *as* 0.5*.*
- What is the decimal expansion of $\frac{142}{6}$?
  - *The fraction* $\frac{142}{6}$ *is equal to the decimal* $23.66666 \ldots$
- Did you need to use the long division algorithm to determine your answer? Why or why not?
  - *Yes, the long division algorithm was necessary because* $\frac{142}{6} = 23 + \frac{2}{3}$ *and* $\frac{2}{3}$ *is not a finite decimal.*
    Note: Some students may have recognized the fraction $\frac{2}{3}$ *as* $0.6666 \ldots$ *and not used the long division algorithm to determine the decimal expansion.*
- How did you know when you could stop dividing?
  - *I knew to stop dividing because the remainder kept repeating. Specifically, when I used the long division algorithm, the number* 40 *kept appearing, and there are* 6 *groups of* 6 *in* 40*, leaving* 4 *as a remainder each time, which became* 40 *when I brought down another* 0*.*
- We represent the decimal expansion of $\frac{142}{6}$ as $23.\overline{6}$, where the line above the 6 is the "repeating block"; that is, the digit 6 repeats as we saw in the long division algorithm.
- What is the decimal expansion of $\frac{142}{11}$?
  - *The fraction* $\frac{142}{11}$ *is equal to the decimal* $12.90909090 \ldots$
- Did you need to use the long division algorithm to determine your answer? Why or why not?
  - *Yes, the long division algorithm was necessary because* $\frac{142}{11} = 12 + \frac{10}{11}$ *and* $\frac{10}{11}$ *is not a finite decimal.*
- How did you know when you could stop dividing?
  - *I knew to stop dividing because the remainder kept repeating. Specifically, when I used the long division algorithm, I kept getting the number* 10*, which is not divisible by* 11*, so I had to bring down another* 0 *making the number* 100*. This kept happening, so I knew to stop once I noticed the work I was doing was the same.*

- Which block of digits kept repeating?
  - *The block of digits that kept repeating was* 90.
- How do we represent the decimal expansion of $\frac{142}{11}$?
  - *The decimal expansion of* $\frac{142}{11}$ *is* $12.\overline{90}$.
- In general, we say that every rational number has a decimal expansion that repeats eventually. It is obvious by the repeat blocks that $\frac{142}{6}$ and $\frac{142}{11}$ are rational numbers. Are the numbers $\frac{142}{2}$ and $\frac{142}{4}$ rational? If so, what is their repeat block?

Provide students a minute or two to discuss in small groups what the repeat blocks for $\frac{142}{2}$ and $\frac{142}{4}$ are.

- *The decimal expansion of* $\frac{142}{2}$ *is* $71.0000\ldots$ *where the repeat block is* 0. *The decimal expansion of* $\frac{142}{4}$ *is* $35.50000\ldots$ *where the repeat block is* 0. *Since the numbers* $\frac{142}{2}$ *and* $\frac{142}{4}$ *have decimal expansions that repeat, then the numbers are rational.*

Exercises 6–10 (5 minutes)

Students complete Exercises 6–10 independently.

Exercises 6–10

6. **Does the number $\frac{65}{13}$ have a finite or infinite decimal expansion? Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

   The number $\frac{65}{13}=\frac{5\times 13}{13}=5$ ***so it is a finite decimal. The decimal expansion of*** $\frac{65}{13}$ ***is*** $5.0000\ldots$ ***where the repeat block is*** 0***. Therefore, the number is rational.***

7. **Does the number $\frac{17}{11}$ have a finite or infinite decimal expansion? Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

   $$\frac{17}{11}=\frac{1\times 11}{11}+\frac{6}{11}$$

   $$\begin{array}{r} 1.5454 \\ 11\overline{)17.00000} \\ \underline{11} \\ 6\,0 \\ \underline{5\,5} \\ 50 \\ \underline{44} \\ 60 \\ \underline{55} \\ 50 \\ \underline{44} \\ 60 \end{array}$$

   The number $\frac{17}{11}$ ***has an infinite decimal expansion,*** $1.\overline{54}$***. The block of digits*** 54 ***repeats. In doing the long division, I realized that the remainder of*** 6 ***and remainder of*** 5 ***kept reappearing in my work. Since the number has a repeat block, it is rational.***

8. **Does the number $\pi = 3.1415926535897\ldots$ have a finite or infinite decimal expansion? Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

The number has an infinite decimal expansion. However, it does not have decimal digits that repeat in a block. For that reason, the number is not rational.

9. **Does the number $\frac{860}{999} = 0.860860860\ldots$ have a finite or infinite decimal expansion? Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

The number has an infinite decimal expansion. However, the decimal expansion has a repeat block of 860. Because every rational number has a block that repeats, the number is rational.

10. **Does the number $\sqrt{2} = 1.41421356237\ldots$ have a finite or infinite decimal expansion? Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

The number has an infinite decimal expansion. However, it does not have decimal digits that repeat in a block. For that reason, the number is not rational.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that the long division algorithm is a procedure that allows us to write the decimal expansion for infinite decimals.
- We know that every rational number has a decimal expansion that repeats eventually.

Lesson Summary

The long division algorithm is a procedure that can be used to determine the decimal expansion of infinite decimals.

Every rational number has a decimal expansion that repeats eventually. For example, the number 32 is rational because it has a repeat block of the digit 0 in its decimal expansion, $32.\overline{0}$. The number $\frac{1}{3}$ is rational because it has a repeat block of the digit 3 in its decimal expansion, $0.\overline{3}$. The number $0.454545\ldots$ is rational because it has a repeat block of the digits 45 in its decimal expansion, $0.\overline{45}$.

Exit Ticket (5 minutes)

Name ______________________________ Date ____________

Lesson 8: The Long Division Algorithm

Exit Ticket

1. Write the decimal expansion of $\frac{125}{8}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

2. Write the decimal expansion of $\frac{13}{7}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

Exit Ticket Sample Solutions

1. **Write the decimal expansion of $\frac{125}{8}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

$$\frac{125}{8} = \frac{15 \times 8}{8} + \frac{5}{8} = 15\frac{5}{8}$$

```
    15.625
 8)125.000
   0
   12
    8
    45
    40
     50
     48
      20
      16
       40
       40
        0
```

The decimal expansion of $\frac{125}{8}$ is 15.625. The number is rational because it is a finite decimal with a repeating block of 0.

2. **Write the decimal expansion of $\frac{13}{7}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

$$\frac{13}{7} = \frac{1 \times 7}{7} + \frac{6}{7} = 1\frac{6}{7}$$

```
    1.857142857142
 7)13.0000000000000
   7
   60
   56
    40
    35
     50
     49
      10
       7
       30
       28
        20
        14
         60
         56
          40
          35
           50
```

The decimal expansion of $\frac{13}{7}$ is $1.\overline{857142}$. The number is rational because there is a repeating block of 857142.

Rational numbers have decimal expansions that repeat; therefore, $\frac{13}{7}$ is a rational number.

Problem Set Sample Solutions

1. **Write the decimal expansion of $\frac{7000}{9}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

$$\frac{7000}{9} = \frac{777 \times 9}{9} + \frac{7}{9} = 777\frac{7}{9}$$

$$\begin{array}{r} 777.77 \\ 9\overline{)7000.00} \\ \underline{63} \\ 70 \\ \underline{63} \\ 70 \\ \underline{63} \\ 70 \\ \underline{63} \\ 70 \\ \underline{63} \\ 70 \end{array}$$

The decimal expansion of $\frac{7000}{9}$ is $777.\overline{7}$. The number is rational because it has the repeating digit of 7. Rational numbers have decimal expansions that repeat; therefore, $\frac{7000}{9}$ is a rational number.

2. **Write the decimal expansion of $\frac{6555555}{3}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

$$\frac{6555555}{3} = \frac{2185185 \times 3}{3} + \frac{0}{3} = 2,185,185$$

The decimal expansion of $\frac{6555555}{3}$ is $2,185,185$. The number is rational because we can write the repeating digit of 0 following the whole number. Rational numbers have decimal expansions that repeat; therefore, $\frac{6555555}{3}$ is a rational number.

3. **Write the decimal expansion of $\frac{350000}{11}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

$$\frac{350000}{11} = \frac{31,818 \times 11}{11} + \frac{2}{11}$$
$$= 31,818\frac{2}{11}$$

$$\begin{array}{r} 31818.18 \\ 11\overline{)350,000.00} \\ \underline{33} \\ 20 \\ \underline{11} \\ 90 \\ \underline{88} \\ 20 \\ \underline{11} \\ 90 \\ \underline{88} \\ 20 \\ \underline{11} \\ 90 \\ \underline{88} \\ 20 \end{array}$$

The decimal expansion of $\frac{350000}{11}$ is $31818.\overline{18}$. The number is rational because there is a repeating block of 18. Rational numbers have decimal expansions that repeat; therefore, $\frac{350000}{11}$ is a rational number.

4. **Write the decimal expansion of $\frac{12000000}{37}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

$$\frac{12,000,000}{37} = \frac{324,324 \times 37}{37} + \frac{12}{37}$$
$$= 324,324\frac{12}{37}$$

$$\begin{array}{r} 324324.324 \\ 37\overline{)12,000,000.000} \\ \underline{111} \\ 90 \\ \underline{74} \\ 160 \\ \underline{148} \\ 120 \\ \underline{111} \\ 90 \\ \underline{74} \\ 160 \\ \underline{148} \\ 120 \\ \underline{111} \\ 90 \\ \underline{74} \\ 160 \\ \underline{148} \\ 120 \end{array}$$

The decimal expansion of $\frac{12000000}{37}$ is $324,324.\overline{324}$. The number is rational because there is a repeating block of 324. Rational numbers have decimal expansions that repeat; therefore, $\frac{12000000}{37}$ is a rational number.

5. Someone notices that the long division of $2,222,222$ by 6 has a quotient of $370,370$ and remainder 2 and wonders why there is a repeating block of digits in the quotient, namely 370. Explain to the person why this happens.

$$\frac{2222222}{6} = \frac{370,370 \times 6}{6} + \frac{2}{6}$$
$$= 370,370\frac{2}{6}$$

```
      370370
   6)2222222
     18
      42
      42
       022
        18
         42
         42
          02
```

The reason that the block of digits 370 keeps repeating is because the long division algorithm leads us to perform the same division over and over again. In the algorithm shown above, we see that there 3 groups of 6 in 22, leaving a remainder of 4. When we bring down the next 2 we see that there are exactly 7 groups of 6 in 42. When we bring down the next 2 we see that there are 0 groups of 6 in 2, leaving a remainder of 2. It is then that the process starts over because the next step is to bring down another 2, giving us 22, which is what we started with. Since the division repeats, then the digits in the quotient will repeat.

6. Is the number $\frac{9}{11} = 0.81818181\ldots$ rational? Explain.

The number appears to be rational because the decimal expansion has a repeat block of 81. Because every rational number has a block that repeats, the number is rational.

7. Is the number $\sqrt{3} = 1.73205080\ldots$ rational? Explain.

The number appears to have a decimal expansion that does not have decimal digits that repeat in a block. For that reason, this is not a rational number.

8. Is the number $\frac{41}{333} = 0.1231231231\ldots$ rational? Explain.

The number appears to be rational because the decimal expansion has a repeat block of 123. Because every rational number has a block that repeats, the number is rational.