



Lesson 10: Converting Repeating Decimals to Fractions

Student Outcomes

- Students know the intuitive reason why every repeating decimal is equal to a fraction. Students convert a decimal expansion that eventually repeats into a fraction.
- Students know that the decimal expansions of rational numbers repeat eventually.
- Students understand that irrational numbers are numbers that are not rational. Irrational numbers cannot be represented as a fraction and have infinite decimals that never repeat.

Classwork

Discussion (4 minutes)

- We have just seen that every fraction (therefore every rational number) is equal to a repeating decimal, and we have learned strategies for determining the decimal expansion of fractions. Now we must learn how to write a repeating decimal as a fraction.
- We begin by noting a simple fact about finite decimals: Given a finite decimal, such as 1.2345678, if we multiply the decimal by 10^5 we get 123,456.78. That is, when we multiply by a power of 10, in this case 10^5 , the decimal point is moved 5 places to the right, i.e.,

$$1.2345678 \times 10^5 = 123,456.78$$

This is true because of what we know about the Laws of Exponents:

$$\begin{aligned} 10^5 \times 1.2345678 &= 10^5 \times (12,345,678 \times 10^{-7}) \\ &= 12,345,678 \times 10^{-2} \\ &= 123,456.78 \end{aligned}$$

- We have discussed in previous lessons that we treat infinite decimals as finite decimals in order to compute with them. For that reason, we will now apply the same basic fact we observed about finite decimals to infinite decimals. That is,

$$1.2345678 \dots \times 10^5 = 123,456.78 \dots$$

We will use this fact to help us write infinite decimals as fractions.

Example 1 (10 minutes)

Example 1

Find the fraction that is equal to the infinite decimal $0.\overline{81}$.

- We want to find the fraction that is equal to the infinite decimal $0.\overline{81}$.
- We let $x = 0.\overline{81}$.



MP.1

Allow students time to work in pairs or small groups to write the fraction equal to $0.\overline{81}$. Students should recognize that the preceding discussion has something to do with this process and should be an entry point for finding the solution. They should also recognize that since we let $x = 0.\overline{81}$, an equation of some form will lead them to the fraction. Give them time to make sense of the problem. Make a plan for finding the fraction, and then attempt to figure it out.

- Since $x = 0.\overline{81}$, we will multiply both sides of the equation by 10^2 and then solve for x . We will multiply by 10^2 because there are two decimal digits that repeat immediately following the decimal point.

$$\begin{aligned} x &= 0.\overline{81} \\ x &= 0.81818181 \dots \\ 10^2x &= (10^2)0.81818181 \dots \\ 100x &= 81.818181 \dots \end{aligned}$$

Ordinarily we would finish solving for x by dividing both sides of the equation by 100. Do you see why that is not a good plan for this problem?

- If we divide both sides by 100, we would get $x = \frac{81.818181\dots}{100}$, which does not really show us that the repeating decimal is equal to a fraction (rational number) because the repeating decimal is still in the numerator.

MP.7

- We know that $81.818181 \dots$ is the same as $81 + 0.818181 \dots$. Then by substitution, we have $100x = 81 + 0.818181 \dots$

How can we rewrite $100x = 81 + 0.818181 \dots$ in a useful way using the fact that $x = 0.\overline{81}$?

- We can rewrite $100x = 81 + 0.818181 \dots$ as $100x = 81 + x$ because x represents the repeating decimal block $0.818181 \dots$

Now we can solve for x to find the fraction that represents the repeating decimal $0.\overline{81}$:

$$\begin{aligned} 100x &= 81 + x \\ 100x - x &= 81 + x - x \\ (100 - 1)x &= 81 \\ 99x &= 81 \\ \frac{99x}{99} &= \frac{81}{99} \\ x &= \frac{81}{99} \\ x &= \frac{9}{11} \end{aligned}$$

- Therefore, the repeating decimal $0.\overline{81} = \frac{9}{11}$.

Have students verify that we are correct using a calculator.

Exercises 1–2 (5 minutes)

Students complete Exercises 1–2 in pairs. Allow students to use a calculator to check their work.

Exercises 1–2

- Let $x = 0.\overline{123}$. Explain why multiplying both sides of this equation by 10^3 will help us determine the fractional representation of x .

When we multiply both sides of the equation by 10^3 , on the right side we will have 123. 123123 This is helpful because we will be able to subtract the repeating decimal from both sides by subtracting x .

- b. After multiplying both sides of the equation by 10^3 , rewrite the resulting equation by making a substitution that will help determine the fractional value of x . Explain how you were able to make the substitution.

$$\begin{aligned}x &= 0.\overline{123} \\10^3x &= (10^3)0.\overline{123} \\1,000x &= 123.\overline{123} \\1,000x &= 123 + 0.123123 \dots \\1,000x &= 123 + x\end{aligned}$$

Since we let $x = 0.\overline{123}$, we can substitute the repeating decimal $0.123123 \dots$ with x .

- c. Solve the equation to determine the value of x .

$$\begin{aligned}1,000x - x &= 123 + x - x \\999x &= 123 \\999x &= \frac{123}{999} \\x &= \frac{123}{999} \\x &= \frac{41}{333}\end{aligned}$$

- d. Is your answer reasonable? Check your answer using a calculator.

Yes, my answer is reasonable and correct. It is reasonable because the denominator cannot be expressed as a product of 2's and 5's; therefore, I know that the fraction must represent an infinite decimal. It is also reasonable because the decimal value is closer to 0 than to 0.5, and the fraction $\frac{41}{333}$ is also closer to 0 than to $\frac{1}{2}$. It is correct because the division of $\frac{41}{333}$ using a calculator is $0.123123 \dots$

2. Find the fraction equal to $0.\overline{4}$. Check that you are correct using a calculator.

Let $x = 0.\overline{4}$

$$\begin{aligned}x &= 0.\overline{4} \\10x &= (10)0.\overline{4} \\10x &= 4.\overline{4} \\10x &= 4 + x \\10x - x &= 4 + x - x \\9x &= 4 \\9x &= \frac{4}{9} \\x &= \frac{4}{9}\end{aligned}$$

Example 2 (6 minutes)

Example 2

Find the fraction that is equal to the infinite decimal $2.13\overline{8}$.

- We want to find the fraction that is equal to the infinite decimal $2.13\overline{8}$. Notice that this time there is just one digit that repeats, but it is three places to the right of the decimal point. If we let $x = 2.13\overline{8}$, by what power of 10 should we multiply? Explain.
 - *The goal is to multiply by a power of 10 so that the only remaining decimal digits are those that repeat. For that reason, we should multiply by 10^2 .*



- We let $x = 2.13\bar{8}$, and multiply both sides of the equation by 10^2 .

$$\begin{aligned}x &= 2.13\bar{8} \\10^2x &= (10^2)2.13\bar{8} \\100x &= 213.\bar{8} \\100x &= 213 + 0.\bar{8}\end{aligned}$$

This time, we cannot simply subtract x from each side. Explain why.

- Subtracting x in previous problems allowed us to completely remove the repeating decimal. This time, $x = 2.13\bar{8}$, not just $0.\bar{8}$.*
- What we will do now is treat $0.\bar{8}$ as a separate, mini-problem. Determine the fraction that is equal to $0.\bar{8}$.
 - Let $y = 0.\bar{8}$.

$$\begin{aligned}y &= 0.\bar{8} \\10y &= 8.\bar{8} \\10y &= 8 + 0.\bar{8} \\10y &= 8 + y \\10y - y &= 8 + y - y \\9y &= 8 \\ \frac{9y}{9} &= \frac{8}{9} \\y &= \frac{8}{9}\end{aligned}$$

- Now that we know that $0.\bar{8} = \frac{8}{9}$, we will go back to our original problem:

$$\begin{aligned}100x &= 213 + 0.\bar{8} \\100x &= 213 + \frac{8}{9} \\100x &= \frac{213 \times 9}{9} + \frac{8}{9} \\100x &= \frac{213 \times 9 + 8}{9} \\100x &= \frac{1925}{9} \\ \frac{1}{100}(100x) &= \frac{1925}{9} \left(\frac{1}{100} \right) \\x &= \frac{1925}{900} \\x &= \frac{77}{36}\end{aligned}$$



Exercises 3–4 (6 minutes)

Students complete Exercises 3–4 independently or in pairs. Allow students to use a calculator to check their work.

Exercises 3–4

3. Find the fraction equal to $1.\overline{623}$. Check that you are correct using a calculator.

| | | |
|--|--|--|
| <p><i>Let $x = 1.\overline{623}$</i></p> $x = 1.\overline{623}$ $10x = (10)1.\overline{623}$ $10x = 16.\overline{23}$ | <p><i>Let $y = 0.\overline{23}$</i></p> $y = 0.\overline{23}$ $10^2y = (10^2)0.\overline{23}$ $100y = 23.\overline{23}$ $100y = 23 + y$ $100y - y = 23 + y - y$ $99y = 23$ $\frac{99y}{99} = \frac{23}{99}$ $y = \frac{23}{99}$ | $10x = 16.\overline{23}$ $10x = 16 + y$ $10x = 16 + \frac{23}{99}$ $10x = \frac{16 \times 99}{99} + \frac{23}{99}$ $10x = \frac{16 \times 99 + 23}{99}$ $10x = \frac{1,607}{99}$ $\frac{1}{10}(10x) = \frac{1}{10}\left(\frac{1,607}{99}\right)$ $x = \frac{1,607}{990}$ |
|--|--|--|

$1.\overline{623} = \frac{1,607}{990}$

4. Find the fraction equal to $2.9\overline{60}$. Check that you are correct using a calculator.

| | | |
|--|--|---|
| <p><i>Let $x = 2.9\overline{60}$</i></p> $x = 2.9\overline{60}$ $10x = (10)2.9\overline{60}$ $10x = 29.\overline{60}$ | <p><i>Let $y = 0.\overline{60}$</i></p> $y = 0.\overline{60}$ $10^2y = (10^2)0.\overline{60}$ $100y = 60.\overline{60}$ $100y = 60 + y$ $100y - y = 60 + y - y$ $99y = 60$ $\frac{99y}{99} = \frac{60}{99}$ $y = \frac{60}{99}$ $y = \frac{20}{33}$ | $10x = 29.\overline{60}$ $10x = 29 + y$ $10x = 29 + \frac{20}{33}$ $10x = \frac{29 \times 33}{33} + \frac{20}{33}$ $10x = \frac{29 \times 33 + 20}{33}$ $10x = \frac{977}{33}$ $\left(\frac{1}{10}\right)10x = \left(\frac{1}{10}\right)\frac{977}{33}$ $x = \frac{977}{330}$ |
|--|--|---|

$2.9\overline{60} = \frac{977}{330}$



Discussion (4 minutes)

- What we have observed so far is that when an infinite decimal repeats, it can be written as a fraction, which means that it is a rational number. Do you think infinite decimals that do not repeat are rational as well? Explain.

Provide students time to discuss with a partner before sharing their thoughts with the class.

- Considering the work from this lesson, it does not seem reasonable that an infinite decimal that does not repeat can be expressed as a fraction. We would not have a value that we could set for x and use to compute in order to find the fraction. For those reasons, we do not believe that an infinite decimal that does not repeat is a rational number.
- Infinite decimals that do not repeat are *irrational numbers*, that is, when a number is not equal to a rational number, it is irrational. What we will learn next is how to use rational approximation to determine the approximate decimal expansion of an irrational number.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that to work with infinite decimals we must treat them as finite decimals.
- We know how to use our knowledge of powers of 10 and linear equations to write an infinite decimal that repeats as a fraction.
- We know that every decimal that eventually repeats is a rational number.

Lesson Summary

Numbers with decimal expansions that repeat are rational numbers and can be converted to fractions using a linear equation.

Example: Find the fraction that is equal to the number $0.\overline{567}$.

Let x represent the infinite decimal $0.\overline{567}$.

| | |
|--|---|
| $x = 0.\overline{567}$ | |
| $10^3x = 10^3(0.\overline{567})$ | Multiply by 10^3 because there are 3 digits that repeat |
| $1000x = 567.\overline{567}$ | Simplify |
| $1000x = 567 + 0.\overline{567}$ | By addition |
| $1000x = 567 + x$ | By substitution; $x = 0.\overline{567}$ |
| $1000x - x = 567 + x - x$ | Subtraction Property of Equality |
| $999x = 567$ | Simplify |
| $\frac{999}{999}x = \frac{567}{999}$ | Division Property of Equality |
| $x = \frac{567}{999} = \frac{63}{111}$ | Simplify |

This process may need to be used more than once when the repeating digits do not begin immediately after the decimal. For numbers such as $1.\overline{26}$, for example.

Irrational numbers are numbers that are not rational. They have infinite decimals that do not repeat and cannot be represented as a fraction.

Exit Ticket (5 minutes)



Name _____

Date _____

Lesson 10: Converting Repeating Decimals to Fractions

Exit Ticket

1. Find the fraction equal to $0.\overline{534}$.

2. Find the fraction equal to $3.0\overline{15}$.

Exit Ticket Sample Solutions

1. Find the fraction equal to $0.\overline{534}$.

Let $x = 0.\overline{534}$.

$$\begin{aligned} x &= 0.\overline{534} \\ 10^3x &= (10^3)0.\overline{534} \\ 1,000x &= 534.\overline{534} \\ 1,000x &= 534 + x \\ 1,000x - x &= 534 + x - x \\ 999x &= 534 \\ \frac{999x}{999} &= \frac{534}{999} \\ x &= \frac{534}{999} \\ x &= \frac{178}{333} \end{aligned}$$

$$0.\overline{534} = \frac{178}{333}$$

2. Find the fraction equal to $3.0\overline{15}$.

Let $x = 3.0\overline{15}$

$$\begin{aligned} x &= 3.0\overline{15} \\ 10x &= (10)3.0\overline{15} \\ 10x &= 30.\overline{15} \end{aligned}$$

$$3.0\overline{15} = \frac{199}{66}$$

Let $y = 0.\overline{15}$

$$\begin{aligned} y &= 0.\overline{15} \\ 10^2y &= (10^2)0.\overline{15} \\ 100y &= 15.\overline{15} \\ 100y &= 15 + y \\ 100y - y &= 15 + y - y \\ 99y &= 15 \\ \frac{99y}{99} &= \frac{15}{99} \\ y &= \frac{5}{33} \end{aligned}$$

$10x = 30.\overline{15}$

$$\begin{aligned} 10x &= 30 + y \\ 10x &= 30 + \frac{5}{33} \\ 10x &= \frac{30 \times 33}{33} + \frac{5}{33} \\ 10x &= \frac{30 \times 33 + 5}{33} \\ 10x &= \frac{995}{33} \\ \frac{1}{10}(10x) &= \frac{1}{10}\left(\frac{995}{33}\right) \\ x &= \frac{995}{330} \\ x &= \frac{199}{66} \end{aligned}$$

Problem Set Sample Solutions

1. a. Let $x = 0.\overline{631}$. Explain why multiplying both sides of this equation by 10^3 will help us determine the fractional representation of x .

When we multiply both sides of the equation by 10^3 , on the right side we will have 631.631631 This is helpful because we will be able to subtract the repeating decimal from both sides by subtracting x .

- b. After multiplying both sides of the equation by 10^3 , rewrite the resulting equation by making a substitution that will help determine the fractional value of x . Explain how you were able to make the substitution.

$$\begin{aligned} x &= 0.\overline{631} \\ 10^3x &= (10^3)0.\overline{631} \\ 1,000x &= 631.\overline{631} \\ 1,000x &= 631 + 0.631631 \dots \\ 1,000x &= 631 + x \end{aligned}$$

Since we let $x = 0.\overline{631}$, we can substitute the repeating decimal 0.631631 ... with x .

- c. Solve the equation to determine the value of x .

$$\begin{aligned} 1,000x - x &= 631 + x - x \\ 999x &= 631 \\ \frac{999x}{999} &= \frac{631}{999} \\ x &= \frac{631}{999} \end{aligned}$$

- d. Is your answer reasonable? Check your answer using a calculator.

Yes, my answer is reasonable and correct. It is reasonable because the denominator cannot be expressed as a product of 2's and 5's; therefore, I know that the fraction must represent an infinite decimal. Also the number 0.631 is closer to 0.5 than 1, and the fraction is also closer to $\frac{1}{2}$ than 1. It is correct because the division $\frac{631}{999}$ using the calculator is 0.631631

2. Find the fraction equal to $3.40\overline{8}$. Check that you are correct using a calculator.

Let $x = 3.40\overline{8}$

$$\begin{aligned} x &= 3.40\overline{8} \\ 10^2x &= (10^2)3.40\overline{8} \\ 100x &= 340.\overline{8} \end{aligned}$$

$$3.40\overline{8} = \frac{767}{225}$$

Let $y = 0.\overline{8}$

$$\begin{aligned} y &= 0.\overline{8} \\ 10y &= 10(0.\overline{8}) \\ 10y &= 8.\overline{8} \\ 10y &= 8 + y \\ 10y - y &= 8 + y - y \\ 9y &= 8 \\ \frac{9y}{9} &= \frac{8}{9} \\ y &= \frac{8}{9} \end{aligned}$$

$100x = 340.\overline{8}$

$$\begin{aligned} 100x &= 340 + y \\ 100x &= 340 + \frac{8}{9} \\ 100x &= \frac{340 \times 9}{9} + \frac{8}{9} \\ 100x &= \frac{340 \times 9 + 8}{9} \\ 100x &= \frac{3,068}{9} \\ \left(\frac{1}{100}\right)100x &= \left(\frac{1}{100}\right)\frac{3,068}{9} \\ x &= \frac{3,068}{900} \\ x &= \frac{767}{225} \end{aligned}$$

3. Find the fraction equal to $0.\overline{5923}$. Check that you are correct using a calculator.

Let $x = 0.\overline{5923}$

$$\begin{aligned} x &= 0.\overline{5923} \\ 10^4x &= (10^4)0.\overline{5923} \\ 10,000x &= 5,923.\overline{5923} \\ 10,000x &= 5,923 + x \\ 10,000x - x &= 5,923 + x - x \\ 9,999x &= 5,923 \\ \frac{9,999x}{9,999} &= \frac{5,923}{9,999} \\ x &= \frac{5,923}{9,999} \end{aligned}$$

4. Find the fraction equal to $2.\overline{382}$. Check that you are correct using a calculator.

Let $x = 2.\overline{382}$

$$\begin{aligned} x &= 2.\overline{382} \\ 10x &= (10)2.\overline{382} \\ 10x &= 23.\overline{82} \end{aligned}$$

$$2.\overline{382} = \frac{2,359}{990}$$

Let $y = 0.\overline{82}$

$$\begin{aligned} y &= 0.\overline{82} \\ 10^2y &= (10^2)0.\overline{82} \\ 100y &= 82.\overline{82} \\ 100y &= 82 + y \\ 100y - y &= 82 + y - y \\ 99y &= 82 \\ \frac{99y}{99} &= \frac{82}{99} \\ y &= \frac{82}{99} \end{aligned}$$

$10x = 23.\overline{82}$

$$\begin{aligned} 10x &= 23 + \frac{82}{99} \\ 10x &= \frac{23 \times 99}{99} + \frac{82}{99} \\ 10x &= \frac{23 \times 99 + 82}{99} \\ 10x &= \frac{2,359}{99} \\ \left(\frac{1}{10}\right)10x &= \left(\frac{1}{10}\right)\frac{2,359}{99} \\ x &= \frac{2,359}{990} \end{aligned}$$

5. Find the fraction equal to $0.\overline{714285}$. Check that you are correct using a calculator.

Let $x = 0.\overline{714285}$.

$$\begin{aligned} x &= 0.\overline{714285} \\ 10^6x &= (10^6)0.\overline{714285} \\ 1,000,000x &= 714,825.\overline{714285} \\ 1,000,000x &= 714,285 + x \\ 1,000,000x - x &= 714,285 + x - x \\ 999,999x &= 714,285 \\ \frac{999,999x}{999,999} &= \frac{714,285}{999,999} \\ x &= \frac{714,285}{999,999} \\ x &= \frac{5}{7} \end{aligned}$$

6. Explain why an infinite decimal that is not a repeating decimal cannot be rational.

Infinite decimals that do repeat can be expressed as a fraction and are therefore rational. The method we learned today to write a repeating decimal as a rational number cannot be applied to infinite decimals that do not repeat. The method requires that we let x represent the repeating part of the decimal. If the number has a decimal expansion that does not repeat, we cannot express the number as a fraction, i.e., a rational number.



7. In a previous lesson we were convinced that it is acceptable to write $0.\overline{9} = 1$. Use what you learned today to show that it is true.

Let $x = 0.\overline{9}$

$$\begin{aligned} x &= 0.\overline{9} \\ 10x &= (10)0.\overline{9} \\ 10x &= 9.\overline{9} \\ 10x &= 9 + x \\ 10x - x &= 9 + x - x \\ 9x &= 9 \\ \frac{9x}{9} &= \frac{9}{9} \\ x &= \frac{9}{9} \\ x &= 1 \end{aligned}$$

8. Examine the following repeating infinite decimals and their fraction equivalents. What do you notice? Why do you think what you observed is true?

$$0.\overline{81} = \frac{81}{99} \quad 0.\overline{4} = \frac{4}{9} \quad 0.\overline{123} = \frac{123}{999} \quad 0.\overline{60} = \frac{60}{99} \quad 0.\overline{9} = 1.0$$

In each case, the fraction that represents the infinite decimal has a numerator that is exactly the repeating part of the decimal and a denominator comprised of 9's. Specifically, the denominator has the same number of digits of 9's as the number of digits that repeat. For example, $0.\overline{81}$ has two repeating decimal digits, so the denominator has two 9's. Since we know that $0.\overline{9} = 1$, we can make the assumption that repeating 9's, like 99 could be expressed as 100, meaning that the fraction $\frac{81}{99}$ is almost $\frac{81}{100}$, which would then be expressed as 0.81.