



## Lesson 11: The Decimal Expansion of Some Irrational Numbers

### Student Outcomes

- Students use rational approximation to get the approximate decimal expansion of numbers like  $\sqrt{3}$  and  $\sqrt{28}$ .
- Students distinguish between rational and irrational numbers based on decimal expansions.

### Lesson Notes

The definition of an irrational number can finally be given and understood completely once students know that the decimal expansion of non-perfect squares like  $\sqrt{3}$  and  $\sqrt{28}$  are infinite and do not repeat. That is, square roots of non-perfect squares cannot be expressed as rational numbers and are therefore defined as irrational numbers.

### Classwork

#### Opening Exercise (5 minutes)

##### Opening Exercise

Place  $\sqrt{28}$  on a number line. What decimal do you think  $\sqrt{28}$  is equal to? Explain your reasoning.

**MP.3**

Lead a discussion where students share their reasoning as to the placement of  $\sqrt{28}$  on the number line. Encourage students to critique the reasoning of others while evaluating their own arguments. Consider having students vote on the placement they think is most correct.

#### Discussion (10 minutes)

- We have studied the properties of rational numbers; today we will finally be able to characterize those numbers that are not rational.
- So far we have been able to estimate the size of a number like  $\sqrt{3}$  by stating that it is between the two perfect squares  $\sqrt{1}$  and  $\sqrt{4}$ , meaning that  $\sqrt{3}$  is between 1 and 2 but closer to 2. In our work so far we have found the decimal expansion of numbers using long division and by inspecting the denominators for products of 2's and 5's. Numbers written with a square root symbol are different and require a different method for determining their decimal expansions. The method we will learn is called rational approximation: using a sequence of rational numbers to get closer and closer to a given number to estimate the value of a number.

**Example 1**

**Example 1**

Recall the Basic Inequality:

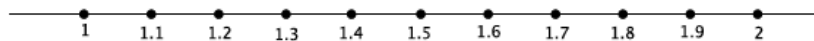
Let  $c$  and  $d$  be two positive numbers, and let  $n$  be a fixed positive integer. Then  $c < d$  if and only if  $c^n < d^n$ .

Write the decimal expansion of  $\sqrt{3}$ .

First approximation:

- We will use the Basic Inequality that we learned in Lesson 3:  
Let  $c$  and  $d$  be two positive numbers, and let  $n$  be a fixed positive integer. Then  $c < d$  if and only if  $c^n < d^n$ .
- To write the decimal expansion of  $\sqrt{3}$  we first determine between which two integers the number  $\sqrt{3}$  would lie on the number line. This is our first approximation. What are those integers?
  - *The number  $\sqrt{3}$  will be between 1 and 2 on the number line because  $1^2 = 1$  and  $2^2 = 4$ .*
- With respect to the Basic Inequality, we can verify that  $\sqrt{3}$  lies between the integers 1 and 2 because  $1^2 < (\sqrt{3})^2 < 2^2$ .
- To be more precise with our estimate of  $\sqrt{3}$ , we now look at the tenths between the numbers 1 and 2. This is our second approximation.

Second approximation:



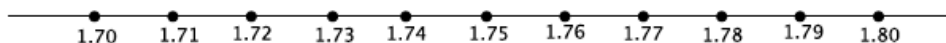
- The question becomes, where exactly would  $\sqrt{3}$  lie on this magnified version of the number line? There are 10 possibilities:  $1.0 < \sqrt{3} < 1.1$ ,  $1.1 < \sqrt{3} < 1.2$ ,  $1.2 < \sqrt{3} < 1.3$ , ..., or  $1.9 < \sqrt{3} < 2.0$ . Use of the Basic Inequality can guide us to selecting the correct possibility. Specifically, we need to determine which of the inequalities shown below is correct:

$$1.0^2 < (\sqrt{3})^2 < 1.1^2, \quad 1.1^2 < (\sqrt{3})^2 < 1.2^2, \quad 1.2^2 < (\sqrt{3})^2 < 1.3^2, \quad \dots, \quad \text{or } 1.9^2 < (\sqrt{3})^2 < 2.0^2.$$

With the help of a calculator we can see that  $1.7^2 < (\sqrt{3})^2 < 1.8^2$  because  $1.7^2 = 2.89$  and  $1.8^2 = 3.24$ ; therefore,  $1.7 < \sqrt{3} < 1.8$ .

- What do you think will need to be done to get an even more precise estimate of the number  $\sqrt{3}$ ?
  - *We will need to look at the interval between 1.7 and 1.8 more closely and repeat the process we did before.*
- Looking at the increments between 1.7 and 1.8, we again have 10 possibilities. This is our third approximation.

Third approximation:





- The Basic Inequality and the help of a calculator show that  $\sqrt{3}$  will be between 1.73 and 1.74 because  $1.73^2 < (\sqrt{3})^2 < 1.74^2$ .

Have students verify using a calculator that  $1.73^2 = 2.9929$  and  $1.74^2 = 3.0276$  and ultimately that  $1.73^2 < (\sqrt{3})^2 < 1.74^2$ .

- What do you think will need to be done to get an even more precise estimate of the number  $\sqrt{3}$ ?
  - *We will need to look at the interval between 1.73 and 1.74 more closely and repeat the process we did before.*
- At this point the pattern should be clear. Now to look more carefully at what we are actually doing. We began by looking at the sequence of integers, specifically between two positive integers 1 and 2. Think of this interval as  $10^0$  (because it equals 1). Then we looked at the sequence of tenths between 1 and 2; think of this interval as  $10^{-1}$  (because it equals  $\frac{1}{10}$ ). Then we looked at the sequence of hundredths between 1.7 and 1.8; think of this interval as  $10^{-2}$  (because it equals  $\frac{1}{100}$ ). To determine the location of  $\sqrt{3}$ , we had to look between points that differ by  $10^{-n}$  for any positive integer  $n$ . The intervals we investigate, i.e.,  $10^{-n}$ , get increasingly smaller as  $n$  gets larger.
- This method of looking at successive intervals is what we call rational approximation. With each new interval we are approximating the value of the number by determining which two rational numbers it lies between.

*Scaffolding:*

Identifying the various forms of approximate, approximately, and approximation, may be useful to English Language Learners.

### Discussion (15 minutes)

The following discussion revisits the Opening Exercise. Before you begin, ask students to reevaluate their own reasoning, and if you had them vote, consider asking them to vote again to see if anyone wants to change their mind about the best approximation for  $\sqrt{28}$ .

### Example 2

**Example 2**

Write the decimal expansion of  $\sqrt{28}$ .

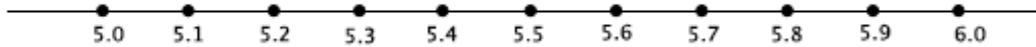
First approximation:

- We will use the method of rational approximation to estimate the location of  $\sqrt{28}$  on the number line.
- What interval of integers, i.e., an interval equal to  $10^0$ , do we examine first? Explain.
  - *We must examine the interval between 5 and 6 because  $5^2 < (\sqrt{28})^2 < 6^2$ , i.e.,  $25 < 28 < 36$ .*



- Now we examine the interval of tenths, i.e.,  $10^{-1}$ , between 5 and 6. Where might  $\sqrt{28}$  lie?

Second approximation:

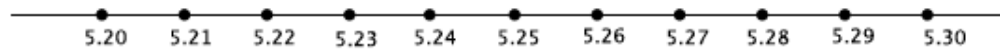


- The number  $\sqrt{28}$  will lie between 5.0 and 5.1 or 5.1 and 5.2 or...5.9 and 6.0.
- How do we determine which interval is correct?
  - We must use the Basic Inequality to check each interval. For example, we need to see if the following inequality is true:  $5.0^2 < (\sqrt{28})^2 < 5.1^2$
- Before we begin checking each interval, let's think about how we can be more methodical in our approach. We know that  $\sqrt{28}$  is between 5 and 6, but which integer is it closer to?
  - The number  $\sqrt{28}$  will be closer to 5 than 6.
- Then we should begin checking the intervals beginning with 5 and work our way up. If the number were closer to 6, then we would begin checking the intervals on the right first and work our way down.

Provide students time to determine which interval the number  $\sqrt{28}$  will lie between. Ask students to share their findings, and then continue the discussion.

- Now that we know that the number  $\sqrt{28}$  lies between 5.2 and 5.3, let's check intervals of hundredths, i.e.,  $10^{-2}$ .

Third approximation:

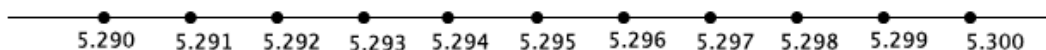


- Again, we should try to be methodical. Since  $5.2^2 = 27.04$  and  $5.3^2 = 28.09$ , where should we begin checking?
  - We should begin with the interval between 5.29 and 5.30 because 28 is closer to 28.09 compared to 27.04.

Provide students time to determine which interval the number  $\sqrt{28}$  will lie between. Ask students to share their findings, and then continue the discussion.

- Now we know that the number  $\sqrt{28}$  is between 5.29 and 5.30. Let's go one step further and examine intervals of thousandths, i.e.,  $10^{-3}$ .

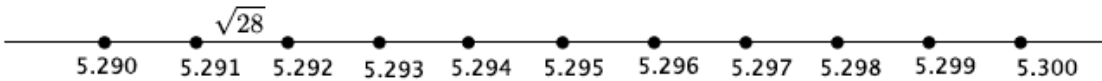
Fourth approximation:



- Since  $5.29^2 = 27.9841$  and  $5.30^2 = 28.09$ , where should we begin our search?
  - We should begin with the interval between 5.290 and 5.291 because 28 is closer to 27.9841 compared to 28.09.

Provide students time to determine which interval the number  $\sqrt{28}$  will lie between. Ask students to share their findings, and then finish the discussion.

- The number  $\sqrt{28}$  lies between 5.291 and 5.292 because  $5.291^2 = 27.994681$  and  $5.292^2 = 28.005264$ . At this point we have a fair approximation of the value of  $\sqrt{28}$ . It is between 5.291 and 5.292 on the number line:



- We could continue this process of rational approximation to see that  $\sqrt{28} = 5.291502622 \dots$ . How is this number different from other infinite decimals we have worked with?
  - *Other infinite decimals we have worked with have a block of digits that repeat at some point. This infinite decimal does not.*
- We know that rational numbers are those that have decimal expansions that eventually repeat. We also know that rational numbers can be expressed as a fraction in the form of a ratio of integers. In the last lesson we learned how to convert a repeating decimal to a fraction. Do you think that same process can be used with a number like  $\sqrt{28} = 5.291502622 \dots$ ?
  - *No because the decimal expansion does not repeat.*

*Scaffolding:*

A graphic organizer may be useful. Consider the one below:

Rational Numbers	Irrational Numbers
Definition:	Definition:
Examples:	Examples:

- Because the number  $\sqrt{28}$  cannot be expressed as a rational number, we say that it is *irrational*. Any number that cannot be expressed as a rational number is, by definition, an irrational number.
- A common irrational number is pi:  $\pi = 3.14159265 \dots$ . Notice that the decimal expansion of pi is infinite and does not repeat. Those qualities are what make pi an irrational number. Often for computations we give pi a rational approximation of 3.14 or  $\frac{22}{7}$ , but those are merely approximations, not the true value of the number pi.
- Another example of an irrational number is  $\sqrt{7}$ . What do you expect the decimal expansion of  $\sqrt{7}$  to look like?
  - *The decimal expansion of  $\sqrt{7}$  will be infinite without a repeating block.*
- The number  $\sqrt{7} = 2.645751311 \dots$ . The decimal expansion is infinite and does not repeat.
- Is the number  $\sqrt{49}$  rational or irrational? Explain.
  - *The number  $\sqrt{49} = 7$ . The decimal expansion of  $\sqrt{49}$  can be written as 7.0000 ... which is an infinite decimal expansion with a repeat block. Therefore,  $\sqrt{49}$  is a rational number.*
- Classify the following numbers as rational or irrational. Be prepared to explain your reasoning.

$$\sqrt{10}, \quad 0.123123123 \dots, \quad \sqrt{64}, \quad \frac{5}{11}$$

Provide students time to classify the numbers. They can do this independently or in pairs. Then select students to share their reasoning. Students should identify  $\sqrt{10}$  as irrational because it has a decimal expansion that can only be approximated by rational numbers. The number 0.123123123 ... is a repeating decimal and can be expressed as a fraction and is therefore rational. The number  $\sqrt{64} = 8$  and is therefore a rational number. The fraction  $\frac{5}{11}$  by definition is a rational number because it is a ratio of integers.

Consider going back to the Opening Exercise to determine whose approximation was the closest.

**Exercise 2 (5 minutes)**

Students work in pairs to complete Exercise 2.

**Exercise 2**

Between which interval of hundredths would  $\sqrt{14}$  be located? Show your work.

*The number  $\sqrt{14}$  is between integers 3 and 4 because  $3^2 < (\sqrt{14})^2 < 4^2$ . Then  $\sqrt{14}$  must be checked for the interval of tenths between 3 and 4. Since  $\sqrt{14}$  is closer to 4, we will begin with the interval from 3.9 to 4.0. The number  $\sqrt{14}$  is between 3.7 and 3.8 because  $3.7^2 = 13.69$  and  $3.8^2 = 14.44$ . Now we must look at the interval of hundredths between 3.7 and 3.8. Since  $\sqrt{14}$  is closer to 3.7, we will begin with the interval 3.70 to 3.71. The number  $\sqrt{14}$  is between 3.74 and 3.75 because  $3.74^2 = 13.9876$  and  $3.75^2 = 14.0625$ .*

**Closing (5 minutes)**

Summarize, or ask students to summarize, the main points from the lesson:

- We know that any number that cannot be expressed as a rational number is an irrational number.
- We know that to determine the approximate value of an irrational number we must determine between which two rational numbers it would lie.
- We know that the method of rational approximation uses a sequence of rational numbers, in increments of  $10^0$ ,  $10^{-1}$ ,  $10^{-2}$ , and so on, to get closer and closer to a given number.
- We have a method for determining the approximate decimal expansion of the square root of an imperfect square, which is an irrational number.

**Lesson Summary**

To get the decimal expansion of a square root of a non-perfect square you must use the method of rational approximation. Rational approximation is a method that uses a sequence of rational numbers to get closer and closer to a given number to estimate the value of the number. The method requires that you investigate the size of the number by examining its value for increasingly smaller powers of 10 (i.e., tenths, hundredths, thousandths, and so on). Since  $\sqrt{22}$  is not a perfect square, you would use rational approximation to determine its decimal expansion.

**Example:**

Begin by determining which two integers the number would lie.

$\sqrt{22}$  is between the integers 4 and 5 because  $4^2 < (\sqrt{22})^2 < 5^2$ , which is equal to  $16 < 22 < 25$ .

Next, determine which interval of tenths the number belongs.

$\sqrt{22}$  is between 4.6 and 4.7 because  $4.6^2 < (\sqrt{22})^2 < 4.7^2$ , which is equal to  $21.16 < 22 < 22.09$ .

Next, determine which interval of hundredths the number belongs.

$\sqrt{22}$  is between 4.69 and 4.70 because  $4.69^2 < (\sqrt{22})^2 < 4.70^2$ , which is equal to  $21.9961 < 22 < 22.09$ .

A good estimate of the value of  $\sqrt{22}$  is 4.69 because 22 is closer to 21.9961 than it is to 22.09.

Notice that with each step we are getting closer and closer to the actual value, 22. This process can continue using intervals of thousandths, ten-thousandths, and so on.

Any number that cannot be expressed as a rational number is called an irrational number. Irrational numbers are those numbers with decimal expansions that are infinite and do not have a repeating block of digits.

**Exit Ticket (5 minutes)**



Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 11: The Decimal Expansion of Some Irrational Numbers

### Exit Ticket

1. Determine the 3 decimal digit approximation of the number  $\sqrt{17}$ .

2. Classify the following numbers as rational or irrational, and explain how you know.

$$\frac{3}{5}, 0.73737373 \dots, \sqrt{31}$$



## Exit Ticket Sample Solutions

1. Determine the 3 decimal digit approximation of the number  $\sqrt{17}$ .

*The number  $\sqrt{17}$  is between integers 4 and 5 because  $4^2 < (\sqrt{17})^2 < 5^2$ . Since  $\sqrt{17}$  is closer to 4, I will start checking the tenths intervals closer to 4.  $\sqrt{17}$  is between 4.1 and 4.2 since  $4.1^2 = 16.81$  and  $4.2^2 = 17.64$ . Checking the hundredths interval,  $\sqrt{17}$  is between 4.12 and 4.13 since  $4.12^2 = 16.9744$  and  $4.13^2 = 17.0569$ . Checking the thousandths interval,  $\sqrt{17}$  is between 4.123 and 4.124 since  $4.123^2 = 16.999129$  and  $4.124^2 = 17.007376$ . Since 17 is closer to  $4.123^2$  than  $4.124^2$ , then the three decimal approximation is approximately 4.123.*

2. Classify the following numbers as rational or irrational, and explain how you know.

$$\frac{3}{5}, 0.73737373 \dots, \sqrt{31}$$

*The number  $\frac{3}{5}$ , by definition, is rational because it is a ratio of integers. The number  $0.73737373 \dots$  is rational because it has a repeat block. For that reason, the number can be expressed as a fraction. The number  $\sqrt{31}$  is irrational because it has a decimal expansion that can only be approximated by rational numbers. That is, the number is not equal to a rational number; therefore, it is irrational.*

## Problem Set Sample Solutions

1. Use the method of rational approximation to determine the decimal expansion of  $\sqrt{84}$ . Determine which interval of hundredths it would lie in.

*The number  $\sqrt{84}$  is between 9 and 10 but closer to 9. Looking at the interval of tenths, beginning with 9.0 to 9.1, the number  $\sqrt{84}$  lies between 9.1 and 9.2 because  $9.1^2 = 82.81$  and  $9.2^2 = 84.64$  but is closer to 9.2. In the interval of hundredths, the number  $\sqrt{84}$  lies between 9.16 and 9.17 because  $9.16^2 = 83.9056$  and  $9.17^2 = 84.0889$ .*

2. Get a 3 decimal digit approximation of the number  $\sqrt{34}$ .

*The number  $\sqrt{34}$  is between 5 and 6 but closer to 6. Looking at the interval of tenths, beginning with 5.9 to 6.0, the number  $\sqrt{34}$  lies between 5.8 and 5.9 because  $5.8^2 = 33.64$  and  $5.9^2 = 34.81$  and is closer to 5.8. In the interval of hundredths, the number  $\sqrt{34}$  lies between 5.83 and 5.84 because  $5.83^2 = 33.9889$  and  $5.84^2 = 34.1056$  and is closer to 5.83. In the interval of thousandths, the number  $\sqrt{34}$  lies between 5.830 and 5.831 because  $5.830^2 = 33.9889$  and  $5.831^2 = 34.000561$  but is closer to 5.831. Since 34 is closer to  $5.831^2$  than  $5.830^2$ , then the 3 decimal digit approximation of the number is approximately 5.831.*

3. Write the decimal expansion of  $\sqrt{47}$  to at least 2 decimal digits.

*The number  $\sqrt{47}$  is between 6 and 7 but closer to 7 because  $6^2 < (\sqrt{47})^2 < 7^2$ . In the interval of tenths, the number  $\sqrt{47}$  is between 6.8 and 6.9 because  $6.8^2 = 46.24$  and  $6.9^2 = 47.61$ . In the interval of hundredths, the number  $\sqrt{47}$  is between 6.85 and 6.86 because  $6.85^2 = 46.9225$  and  $6.86^2 = 47.0596$ . Therefore, to 2 decimal digits, the number  $\sqrt{47}$  is approximately 6.85 but when rounded will be approximately 6.86 because  $\sqrt{47}$  is closer to 6.86 but not quite 6.86.*

4. Write the decimal expansion of  $\sqrt{46}$  to at least 2 decimal digits.

*The number  $\sqrt{46}$  is between integers 6 and 7 because  $6^2 < (\sqrt{46})^2 < 7^2$ . Since  $\sqrt{46}$  is closer to 7, I will start checking the tenths intervals between 6.9 and 7.  $\sqrt{46}$  is between 6.7 and 6.8 since  $6.7^2 = 44.89$  and  $6.8^2 = 46.24$ . Checking the hundredths interval,  $\sqrt{46}$  is between 6.78 and 6.79 since  $6.78^2 = 45.9684$  and  $6.79^2 = 46.1041$ . Since 46 is closer to  $6.78^2$  than  $6.79^2$ , then the two decimal approximation is 6.78.*

5. Explain how to improve the accuracy of decimal expansion of an irrational number.

*In order to improve the accuracy of the decimal expansion of an irrational number, you must examine increasingly smaller increments on the number line. Specifically, increments of decreasing powers of 10. The Basic Inequality allows us to determine which interval a number will be between. We begin by determining which two integers the number lies between and then decrease the power of 10 to look at the interval of tenths. Again using the Basic Inequality, we can narrow down the approximation to a specific interval of tenths. Then we look at the interval of hundredths and use the Basic Inequality to determine which interval of hundredths the number would lie between. Then we examine the interval of thousandths. Again the Basic Inequality allows us to narrow down the approximation to thousandths. The more intervals that are examined, the more accurate the decimal expansion of an irrational number will be.*

6. Is the number  $\sqrt{125}$  rational or irrational? Explain.

*The number  $\sqrt{125}$  is an irrational number because it has a decimal expansion that is infinite and does not repeat. That is, the number  $\sqrt{125}$  cannot be expressed as a rational number; therefore, it is irrational.*

7. Is the number 0.646464646 ... rational or irrational? Explain.

*The number  $0.646464646 \dots = \frac{64}{99}$ ; therefore, it is a rational number. Not only is the number  $\frac{64}{99}$  a quotient of integers, but its decimal expansion is infinite with a repeating block of digits.*

8. Is the number 3.741657387 ... rational or irrational? Explain.

*The number 3.741657387 ... is an irrational number because it has a decimal expansion that is infinite and does not repeat. That is, the number 3.741657387 ... cannot be expressed as a rational number; therefore, it is irrational.*

9. Is the number  $\sqrt{99}$  rational or irrational? Explain.

*The number  $\sqrt{99}$  is an irrational number because it has a decimal expansion that is infinite and does not repeat. That is, the number  $\sqrt{99}$  cannot be expressed as a rational number; therefore, it is irrational.*

10. Challenge: Get a 2 decimal digit approximation of the number  $\sqrt[3]{9}$ .

*The number  $\sqrt[3]{9}$  is between integers 2 and 3 because  $2^3 < (\sqrt[3]{9})^3 < 3^3$ . Since  $\sqrt[3]{9}$  is closer to 2, I will start checking the tenths intervals between 2 and 3.  $\sqrt[3]{9}$  is between 2 and 2.1 since  $2^3 = 8$  and  $2.1^3 = 9.261$ . Checking the hundredths interval,  $\sqrt[3]{9}$  is between 2.08 and 2.09 since  $2.08^3 = 8.998912$  and  $2.09^3 = 9.129329$ . Since 9 is closer to  $2.08^3$  than  $2.09^3$ , the two decimal approximation is 2.08.*