Lesson 13: Comparison of Irrational Numbers

Student Outcomes

- Students use rational approximations of irrational numbers to compare the size of irrational numbers.
- Students place irrational numbers in their approximate locations on a number line.

Classwork

Exploratory Challenge Exercises 1–11 (30 minutes)

Students work in pairs to complete Exercises 1–11. The first exercise may be used to highlight the process of answering and explaining the solution to each question. An emphasis should be placed on students' ability to explain their reasoning. Consider allowing students to use a calculator to check their work, but all decimal expansions should be done by hand. At the end of the Exploratory Challenge, consider asking students to state or write a description of their approach to solving each exercise.

MP.1 & MP.3

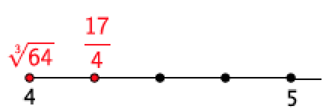
Exercises 1–11

1. **Rodney thinks that $\sqrt[3]{64}$ is greater than $\frac{17}{4}$. Sam thinks that $\frac{17}{4}$ is greater. Who is right and why?**

$\sqrt[3]{64} = \sqrt[3]{4^3} = 4$

$$\frac{17}{4} = \frac{16}{4} + \frac{1}{4}$$
$$= 4 + \frac{1}{4}$$
$$= 4\frac{1}{4}$$

Because $4 < 4\frac{1}{4}$, then $\sqrt[3]{64} < \frac{17}{4}$. So, $\sqrt[3]{64}$ is smaller. The number $\frac{17}{4}$ is equivalent to the mixed number $4\frac{1}{4}$. The cube root of 64 is the whole number 4. Because $4\frac{1}{4}$ is to the right of 4 on the number line, then $4\frac{1}{4}$ is greater than 4, which means that $\frac{17}{4} > \sqrt[3]{64}$; therefore, Sam is correct.



2. Which number is smaller, $\sqrt[3]{27}$ or 2.89? Explain.

$$\sqrt[3]{27} = \sqrt[3]{3^3} = 3$$

Because $2.89 < 3$, then $2.89 < \sqrt[3]{27}$; so, 2.89 is smaller. On a number line, 3 is to the right of 2.89, meaning that 3 is greater than 2.89. Therefore, $2.89 < \sqrt[3]{27}$.

3. Which number is smaller, $\sqrt{121}$ or $\sqrt[3]{125}$? Explain.

$$\sqrt{121} = \sqrt{11^2} = 11$$

$$\sqrt[3]{125} = \sqrt[3]{5^3} = 5$$

Because $5 < 11$, then $\sqrt[3]{125} < \sqrt{121}$. So, $\sqrt[3]{125}$ is smaller. On a number line, the number 5 is to the left of 11, meaning that 5 is less than 11. Therefore, $\sqrt[3]{125} < \sqrt{121}$.

4. Which number is smaller, $\sqrt{49}$ or $\sqrt[3]{216}$? Explain.

$$\sqrt{49} = \sqrt{7^2} = 7$$

$$\sqrt[3]{216} = \sqrt[3]{6^3} = 6$$

Because $6 < 7$, then $\sqrt[3]{216} < \sqrt{49}$. So, $\sqrt[3]{216}$ is smaller. On the number line, 7 is to the right of 6, meaning that 7 is greater than 6. Therefore, $\sqrt[3]{216} < \sqrt{49}$.

5. Which number is greater, $\sqrt{50}$ or $\frac{319}{45}$? Explain.

Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

The number $\frac{319}{45}$ is equal to $7.0\overline{8}$.

The number $\sqrt{50}$ is between 7 and 8 because $7^2 < \left(\sqrt{50}\right)^2 < 8^2$. The number $\sqrt{50}$ is between 7.0 and 7.1 because $7^2 < \left(\sqrt{50}\right)^2 < 7.1^2$. The number $\sqrt{50}$ is between 7.07 and 7.08 because $7.07^2 < \left(\sqrt{50}\right)^2 < 7.08^2$. The approximate decimal value of $\sqrt{50}$ is 7.07. Since $7.07 < 7.0\overline{8}$, then $\sqrt{50} < \frac{319}{45}$; therefore, the fraction $\frac{319}{45}$ is greater.

6. Which number is greater, $\frac{5}{11}$ or $0.\overline{4}$? Explain.

Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

The number $\frac{5}{11}$ is equal to $0.\overline{45}$. Since $0.44444\ldots < 0.454545\ldots$, then $0.\overline{4} < \frac{5}{11}$; therefore, the fraction $\frac{5}{11}$ is greater.

7. Which number is greater, $\sqrt{38}$ or $\frac{154}{25}$? Explain.

Note that students may have used long division or equivalent fractions to determine the decimal expansion of the fraction.

$$\frac{154}{25}=\frac{154\times 4}{25\times 4}=\frac{616}{10^2}=6.16$$

The number $\sqrt{38}$ *is between* 6 *and* 7 *because* $6^2<\left(\sqrt{38}\right)^2<7^2$. *The number* $\sqrt{38}$ *is between* 6.1 *and* 6.2 *because* $6.1^2<\left(\sqrt{38}\right)^2<6.2^2$. *The number* $\sqrt{38}$ *is between* 6.16 *and* 6.17 *because* $6.16^2<\left(\sqrt{38}\right)^2<6.17^2$. *Since* $\sqrt{38}$ *is greater than* 6.16, *then* $\sqrt{38}$ *is greater than* $\frac{154}{25}$.

8. Which number is greater, $\sqrt{2}$ or $\frac{15}{9}$? Explain.

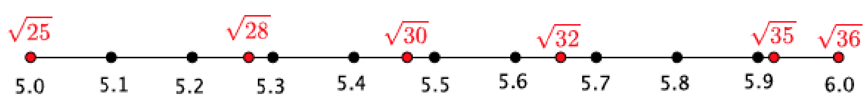
Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

The number $\frac{15}{9}$ *is equal to* $1.\overline{6}$.

The number $\sqrt{2}$ *is between* 1 *and* 2 *because* $1^2<\left(\sqrt{2}\right)^2<2^2$. *The number* $\sqrt{2}$ *is between* 1.4 *and* 1.5 *because* $1.4^2<\left(\sqrt{2}\right)^2<1.5^2$. *Therefore,* $\sqrt{2}<\frac{15}{9}$; *the fraction* $\frac{15}{9}$ *is greater.*

9. Place the following numbers at their approximate location on the number line: $\sqrt{25}$, $\sqrt{28}$, $\sqrt{30}$, $\sqrt{32}$, $\sqrt{35}$, $\sqrt{36}$.

Solutions shown in red.



The number $\sqrt{25}=\sqrt{5^2}=5$.

The number $\sqrt{28}$ *is between* 5 *and* 6. *The number* $\sqrt{28}$ *is between* 5.2 *and* 5.3 *because* $5.2^2<\left(\sqrt{28}\right)^2<5.3^2$.
The number $\sqrt{30}$ *is between* 5 *and* 6. *The number* $\sqrt{30}$ *is between* 5.4 *and* 5.5 *because* $5.4^2<\left(\sqrt{30}\right)^2<5.5^2$.
The number $\sqrt{32}$ *is between* 5 *and* 6. *The number* $\sqrt{32}$ *is between* 5.6 *and* 5.7 *because* $5.6^2<\left(\sqrt{32}\right)^2<5.7^2$.
The number $\sqrt{35}$ *is between* 5 *and* 6. *The number* $\sqrt{35}$ *is between* 5.9 *and* 6.0 *because* $5.9^2<\left(\sqrt{35}\right)^2<6^2$.

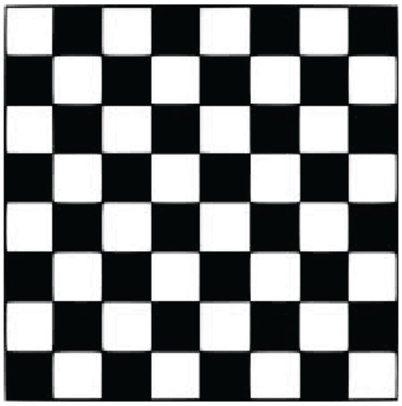
The number $\sqrt{36}=\sqrt{6^2}=6$.

10. Challenge: Which number is larger $\sqrt{5}$ or $\sqrt[3]{11}$?

The number $\sqrt{5}$ *is between* 2 *and* 3 *because* $2^2<\left(\sqrt{5}\right)^2<3^2$. *The number* $\sqrt{5}$ *is between* 2.2 *and* 2.3 *because* $2.2^2<\left(\sqrt{5}\right)^2<2.3^2$. *The number* $\sqrt{5}$ *is between* 2.23 *and* 2.24 *because* $2.23^2<\left(\sqrt{5}\right)^2<2.24^2$. *The number* $\sqrt{5}$ *is between* 2.236 *and* 2.237 *because* $2.236^2<\left(\sqrt{5}\right)^2<2.237^2$. *The decimal expansion of* $\sqrt{5}$ *is approximately* $2.236\ldots$.

The number $\sqrt[3]{11}$ *is between* 2 *and* 3 *because* $2^3<\left(\sqrt[3]{11}\right)^3<3^3$. *The number* $\sqrt[3]{11}$ *is between* 2.2 *and* 2.3 *because* $2.2^3<\left(\sqrt[3]{11}\right)^3<2.3^3$. *The number* $\sqrt[3]{11}$ *is between* 2.22 *and* 2.23 *because* $2.22^3<\left(\sqrt[3]{11}\right)^3<2.23^3$. *The decimal expansion of* $\sqrt[3]{11}$ *is approximately* $2.22\ldots$ *Since* $2.22\ldots<2.236\ldots$, *then* $\sqrt[3]{11}<\sqrt{5}$; *therefore,* $\sqrt{5}$ *is larger.*

11. A certain chessboard is being designed so that each square has an area of 3 in^2. What is the length, rounded to the tenths place, of one edge of the board? (A chessboard is composed of 64 squares as shown.)



The area of one square is 3 in^2. So, if x is the length of one side of one square,

$$x^2 = 3$$
$$\sqrt{x^2} = \sqrt{3}$$
$$x = \sqrt{3}$$

There are 8 squares along one edge of the board, so the length of one edge is $8 \times \sqrt{3}$. The number $\sqrt{3}$ is between 1 and 2 because $1^2 < (\sqrt{3})^2 < 2^2$. The number $\sqrt{3}$ is between 1.7 and 1.8 because $1.7^2 < (\sqrt{3})^2 < 1.8^2$. The number $\sqrt{3}$ is between1.73 and 1.74 because $1.73^2 < (\sqrt{3})^2 < 1.74^2$. The number $\sqrt{3}$ is approximately 1.73. So, the length of one edge of the chessboard is $8 \times 1.73 = 13.84 \approx 13.8$ in.

Note: Some students may determine the total area of the board, $64 \times 3 = 192$, then determine the approximate value of $\sqrt{192} \approx 13.8$, to answer the question.

Discussion (5 minutes)

- How do we know if a number is rational or irrational?
  - *Numbers that can be expressed as a fraction, i.e., a ratio of integers, are by definition rational numbers. Any number that is not rational is irrational.*
- Is the number $1.\overline{6}$ rational or irrational? Explain.
  - *The number $1.\overline{6}$ is rational because it is equal to $\frac{15}{9}$.*
- Is the number $\sqrt{2}$ rational or irrational? Explain.
  - *Since $\sqrt{2}$ is not a perfect square, then $\sqrt{2}$ is an irrational number. This means that the decimal expansion can only be approximated by rational numbers.*
- Which strategy do you use to write the decimal expansion of a fraction? What strategy do you use to write the decimal expansion of square and cube roots?
  - *Student responses will vary. Students will likely state that they use long division or equivalent fractions to write the decimal expansion of fractions. Students will say that they have to use the definition of square and cube roots or rational approximation to write the decimal expansion of the square and cube roots.*

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- The decimal expansion of rational numbers that are expressed as fractions can be found by using long division, by using what we know about equivalent fractions for finite decimals, or by using rational approximation.
- The approximate decimal expansions of irrational numbers (square roots of imperfect squares and imperfect cubes) can be found using rational approximation.
- Numbers, of any form (e.g., fraction, decimal, square root), can be ordered and placed in their approximate location on a number line.

Lesson Summary

The decimal expansion of rational numbers can be found by using long division, equivalent fractions, or the method of rational approximation.

The decimal expansion of irrational numbers can be found using the method of rational approximation.

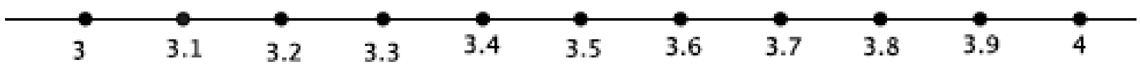
Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 13: Comparison of Irrational Numbers

Exit Ticket

Place the following numbers at their approximate location on the number line: $\sqrt{12}$, $\sqrt{16}$, $\frac{20}{6}$, $3.\overline{53}$, $\sqrt[3]{27}$.



Exit Ticket Sample Solutions

Place the following numbers at their approximate location on the number line: $\sqrt{12}, \sqrt{16}, \frac{20}{6}, 3.\overline{53}, \sqrt[3]{27}$.

Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

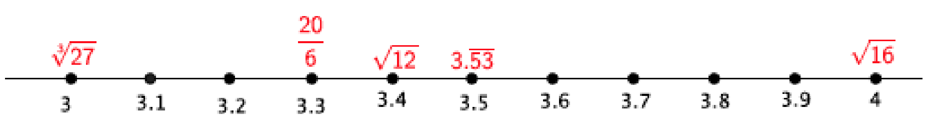
The number $\sqrt{12}$ is between 3.4 and 3.5, since $3.4^2 < (\sqrt{12})^2 < 3.5^2$.

The number $\sqrt{16} = \sqrt{4^2} = 4$.

The number $\frac{20}{6}$ is equal to $3.\overline{3}$.

The number $\sqrt[3]{27} = \sqrt[3]{3^3} = 3$.

Solutions in red:



Problem Set Sample Solutions

1. **Which number is smaller, $\sqrt[3]{343}$ or $\sqrt{48}$? Explain.**

$$\sqrt[3]{343} = \sqrt[3]{7^3} = 7$$

The number $\sqrt{48}$ is between 6 and 7, but definitely less than 7. Therefore, $\sqrt{48} < \sqrt[3]{343}$ and $\sqrt{48}$ is smaller.

2. **Which number is smaller, $\sqrt{100}$ or $\sqrt[3]{1000}$? Explain.**

$$\sqrt{100} = \sqrt{10^2} = 10$$

$$\sqrt[3]{1,000} = \sqrt[3]{10^3} = 10$$

The numbers $\sqrt{100}$ and $\sqrt[3]{1,000}$ are equal because both are equal to 10.

3. **Which number is larger, $\sqrt{87}$ or $\frac{929}{99}$? Explain.**

Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

The number $\frac{929}{99}$ is equal to $9.\overline{38}$.

The number $\sqrt{87}$ is between 9 and 10 because $9^2 < \left(\sqrt{87}\right)^2 < 10^2$. The number $\sqrt{87}$ is between 9.3 and 9.4 because $9.3^2 < \left(\sqrt{87}\right)^2 < 9.4^2$. The number $\sqrt{87}$ is between 9.32 and 9.33 because $9.32^2 < \left(\sqrt{87}\right)^2 < 9.33^2$. The approximate decimal value of $\sqrt{87}$ is $9.32\ldots$. Since $9.32 < 9.\overline{38}$, then $\sqrt{87} < \frac{929}{99}$; therefore, the fraction $\frac{929}{99}$ is larger.

4. Which number is larger, $\frac{9}{13}$ or $0.\overline{692}$? Explain.

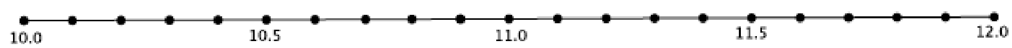
Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

The number $\frac{9}{13}$ is equal to $0.\overline{692307}$. Since $0.692307\ldots < 0.692692\ldots$, then $\frac{9}{13} < 0.\overline{692}$; therefore, the decimal $0.\overline{692}$ is larger.

5. Which number is larger, 9.1 or $\sqrt{82}$? Explain.

The number $\sqrt{82}$ is between 9 and 10 because $9^2 < \left(\sqrt{82}\right)^2 < 10^2$. The number $\sqrt{82}$ is between 9.0 and 9.1 because $9.0^2 < \left(\sqrt{82}\right)^2 < 9.1^2$. Since $\sqrt{82} < 9.1$, then the number 9.1 is larger than the number $\sqrt{82}$.

6. Place the following numbers at their approximate location on the number line: $\sqrt{144}$, $\sqrt[3]{1000}$, $\sqrt{130}$, $\sqrt{110}$, $\sqrt{120}$, $\sqrt{115}$, $\sqrt{133}$. Explain how you knew where to place the numbers.



Solutions shown in red

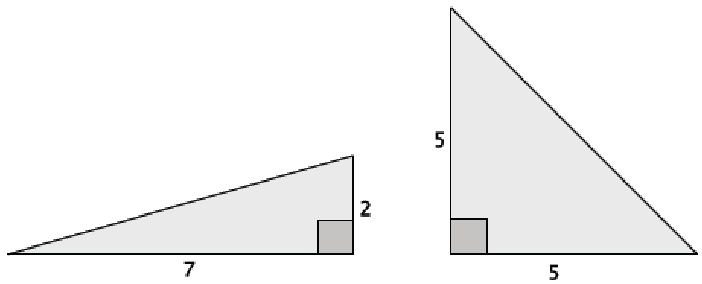
The number $\sqrt{144} = \sqrt{12^2} = 12$.

The number $\sqrt[3]{1000} = \sqrt[3]{10^3} = 10$.

The numbers $\sqrt{110}$, $\sqrt{115}$, and $\sqrt{120}$ are all between 10 and 11 because when squared, their value falls between 10^2 and 11^2. The number $\sqrt{110}$ is between 10.4 and 10.5 because $10.4^2 < \left(\sqrt{110}\right)^2 < 10.5^2$. The number $\sqrt{115}$ is between 10.7 and 10.8 because $10.7^2 < \left(\sqrt{115}\right)^2 < 10.8^2$. The number $\sqrt{120}$ is between 10.9 and 11 because $10.9^2 < \left(\sqrt{120}\right)^2 < 11^2$.

The numbers $\sqrt{130}$ and $\sqrt{133}$ are between 11 and 12 because when squared, their value falls between 11^2 and 12^2. The number $\sqrt{130}$ is between 11.4 and 11.5 because $11.4^2 < \left(\sqrt{130}\right)^2 < 11.5^2$. The number $\sqrt{133}$ is between 11.5 and 11.6 because $11.5^2 < \left(\sqrt{133}\right)^2 < 11.6^2$.

7. Which of the two right triangles shown below, measured in units, has the longer hypotenuse? Approximately how much longer is it?



Let x represent the hypotenuse of the triangle on the left.

$$7^2 + 2^2 = x^2$$
$$49 + 4 = x^2$$
$$53 = x^2$$
$$\sqrt{53} = \sqrt{x^2}$$
$$\sqrt{53} = x$$

The number $\sqrt{53}$ is between 7 and 8 because $7^2 < (\sqrt{53})^2 < 8^2$. The number $\sqrt{53}$ is between 7.2 and 7.3 because $7.2^2 < (\sqrt{53})^2 < 7.3^2$. The number $\sqrt{53}$ is between 7.28 and 7.29 because $7.28^2 < (\sqrt{53})^2 < 7.29^2$. The approximate decimal value of $\sqrt{53}$ is 7.28 ….

Let y represent the hypotenuse of the triangle on the right.

$$5^2 + 5^2 = y^2$$
$$25 + 25 = y^2$$
$$50 = y^2$$
$$\sqrt{50} = \sqrt{y^2}$$
$$\sqrt{50} = y$$

The number $\sqrt{50}$ is between 7 and 8 because $7^2 < (\sqrt{50})^2 < 8^2$. The number $\sqrt{50}$ is between 7.0 and 7.1 because $7.0^2 < (\sqrt{50})^2 < 7.1^2$. The number $\sqrt{50}$ is between 7.07 and 7.08 because $7.07^2 < (\sqrt{50})^2 < 7.08^2$. The approximate decimal value of $\sqrt{50}$ is 7.07 ….

The triangle on the left has the longer hypotenuse. It is approximately 0.21 units longer than the hypotenuse of the triangle on the right.

Note: Based on their experience, some students may reason that $\sqrt{50} < \sqrt{53}$. To answer completely, students must determine the decimal expansion to approximate how much longer one hypotenuse is than the other.