Here's a game I like plying with students. I'll write a positive integer on the board that comes from a set S. You can propose other numbers, and I tell you if your proposed number comes from the set. Eventually you will figure out the properties that define the set. Later in our work, we'll count the number of members of the set. For example, I might write 4561 and put a check mark next to it. You could say, what about 567, and I would say 'no'. You might say, what about 7892, and I would say 'no' again. Then you might guess 2345 and I say 'yes'. Eventually, you conclude that I must be thinking about the set of four digit numbers at include at least one 5.

1. Four digit numbers that include the digit 5.
2. Four digit numbers that use just the digits 6, 7, and 8.
3. Five digit numbers which don't include the digit 0 and have five different digits.
4. Five digit numbers that have no repeated digits.
5. Numbers (any length) that use all different digits.
6. Three digit numbers for which the sum of the digits is 9.
7. Four digit numbers for which the digits increase from left to right, sometimes called *rising numbers*.
8. Four digit numbers for which the numbers do not decrease, sometimes called *non-decreasing numbers*.
9. Six digit numbers $a_1a_2a_3a_4a_5a_6$ that use just the digits 1 through 6, each just once and satisfy the requirement $a_1 < a_2 > a_3 < a_4 > a_5 < a_6$. These are called *up-down numbers*.

We're going to start with some easier problems. For convenience we are discussing numbers that can be built without the digit 0. In other words, we can use only digits from $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. These items are counted in the chart below.

1. How many four-digit number are there?
2. How many four-digit numbers have four different digits?

3. How many four-element subsets does the nine-element set $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ have? Alternatively, how many four-digit increasing numbers are there?

4. How many how many four-digit non-decreasing numbers are there?

Throughout we use both the notations $\binom{n}{r}$ and C_r^n for the number $\frac{n!}{(n-r)!r!}$. We are ready to discuss the general idea of counting samples taken from a population of objects. In doing such sampling we are allowed to make a distinction between the order in which the objects became a part of the sample or not. We are also allowed to sample with replacement or not. This leads to four different types of samples. If we count as different two samples that have the same elements but in a different order, we call these *arrangements*, and if we don't distinguish on this basis, we call the samples *selections*. Let's classify each of the counting problems above using these two questions.

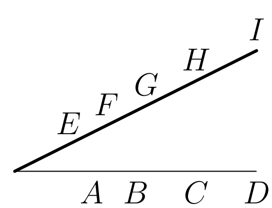
	order matters arrangements()	order does not matter selections{}
with repetitions	Exponations $E_r^n = n^r$ all $9^4 = 6561$ four-digit numbers	Yahtzee Rolls $Y_r^n = C_r^{n+r-1} = \binom{n+r-1}{r}$ all $\binom{9+4-1}{4} = 495$ four-digit nondecreasing numbers
without repetitions	Permutations $P_r^n = \frac{n!}{(n-r)!}$ all $P_4^9 = 9!/5! = 3024$ four-digit numbers with four different digits	Combinations $C_r^n = \frac{n!}{(n-r)!r!} = \binom{n}{r}$ all $\binom{9}{4} = 126$ four-digit increasing numbers

1. (1998 state math contest) There are 8 girls and 6 boys in the Math Club at Central High School. The Club needs to form a delegation to send to a conference, and the delegation must contain exactly two girls and two boys. How many delegations that can be formed?

2. (1998 state math contest) Given n a positive integer, a plus or minus sign is assigned randomly to each of the integers $1, 2, \ldots, n$. Let $P(n)$ be the probability that the sum of the n signed numbers is positive. What is the value of $[P(1) + P(2) + \cdots + P(6)]$?

3. (1999 state math contest) A box contains b red, $2b$ white and $3b$ blue balls, where b is a positive integer. Three balls are selected at random and without replacement from the box. Let $p(b)$ denote the probability that no two of the selected balls have the same color. Is there a value of b for which $p(b) = 1/6$?

4. (2009 Mathcounts) Four points A, B, C and D on one line segment are jointed by line segments to each of five points E, F, G, H, and I on a second line segment. What is the maximum number of points **interior** to the angle belonging to two of these twenty segments.



5. (Mathcounts 2009) How many three-digit numbers can be built from the digits in the list $2, 3, 5, 5, 5, 6, 6$?

6. A *falling* number is an integer whose decimal representation has the property that each digit except the units digit is larger than the one to its right. For example, 96520 is a falling number but 89642 is not. How many five-digit falling numbers are there? How many n-digit falling numbers are there, for $n = 1, 2, 3, 4, 5, 6, 7, 8,$ and 9? What is the total number of falling numbers of all sizes?

7. Cyprian writes down the middle number in each of the $\binom{9}{5} = 126$ five-element subsets of $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then he adds all these numbers together. What sum does he get?

8. Counting sums of subset members.

   (a) How many numbers can be expressed as a sum of two or more distinct members of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$?

   (b) How many integers can be expressed as a sum of two or more different members of the set $\{0, 1, 2, 4, 8, 16, 32\}$?

   (c) How many numbers can be expressed as a sum of four distinct members of the set $\{17, 21, 25, 29, 33, 37, 41\}$?

   (d) How many numbers can be expressed as a sum of two or more distinct members of the set $\{17, 21, 25, 29, 33, 37, 41\}$?

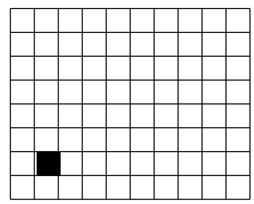
   (e) How many integers can be expressed as a sum of two or more distinct elements of the set $\{1, -3, 9, -27, 81, -243\}$?

9. How many of the first 242 positive integers are expressible as a sum of three or fewer members of the set $\{3^0, 3^1, 3^2, 3^3, 3^4\}$ if we are allowed to use the same power more than once? For example, $5 = 3 + 1 + 1$ can be represented, but 8 cannot. Hint: think about the ternary representations.

10. John has 2 pennies, 3 nickels, 2 dimes, 3 quarters, and 8 dollars. For how many different amounts can John make an exact purchase (with no change required)?

11. How many positive integers less than 1000 have an odd number of positive integer divisors?

12. (2004 AMC 10 and extensions) An 8×10 grid of squares with one shaded square is given.



   (a) How many different squares are bounded by the gridlines?

   (b) How many different rectangles are bounded by the gridlines?

   (c) ($*$)How many different squares bounded by the gridlines contain the shaded square?

   (d) How many different rectangles bounded by the gridlines contain the shaded square?

13. How many squares in the plane have two or more vertices in the set $S = (0,0), (0,1), (0,2), (1,0), (1,1), (2,0)$?

14. Numbers with a given digit sum.

   (a) How many numbers in the set $\{100, 101, 102, \ldots, 999\}$ have a sum of digits equal to 9?

   (b) How many four digit numbers have a sum of digits 9?

   (c) How many integers less than one million have a sum of digits equal to 9?

15. (2008 state math contest) Two coins are removed randomly and without replacement from a box containing 3 nickels, 2 dimes, and 1 coin of value 0. What is the probability that one of the removed coins is worth five cents more than the other?

16. (2008 state math contest) A *palindrome* on the alphabet $\{H, T\}$ is a sequence of h's and T's which reads the same from left to right as from right to left. Thus $HTH, HTTH, HTHTH$ and $HTHHTH$ are palindromes of lengths 3,4,5, and 6 respectively. Let $P(n)$ denote the number of palindromes of length n over $\{H, T\}$. For how many values of n is $1000 < P(n) < 10000$?

17. An urn contains marbles of four colors: red, white, blue, and green. When four marbles are drawn without replacement, the following events are equally likely:

    (a) the selection of four red marbles;

    (b) the selection of one white and three red marbles;

    (c) the selection of one white, one blue, and two red marbles; and

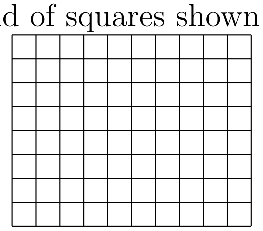
    (d) the selection of one marble of each color.

    What is the smallest number of marbles that the urn could contain?

18. Look at the $m \times n$ multiplication table below. What is the sum of the mn entries in the table?

$\times$	1	2	3	4	5	6	7	8	9	$\ldots$	n
1	1	2	3								n
2	2	4	6								$2n$
3	3	6	9								$3n$
$\vdots$											
m	m	$2m$	$3m$								mn

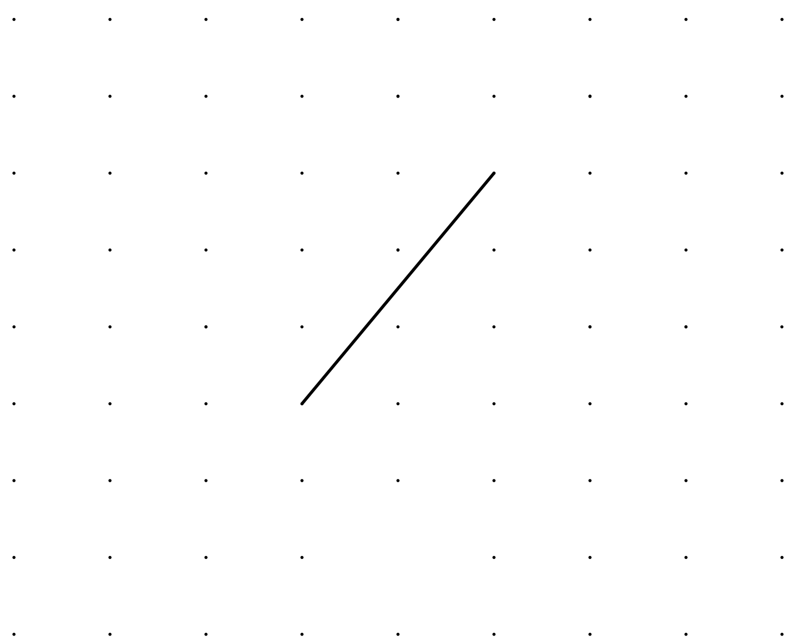
19. Consider the $m \times n$ grid of squares shown below.



    How many rectangles are bounded by the gridlines?

20. How many circles in the plane contain at least three of the points $(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)$?

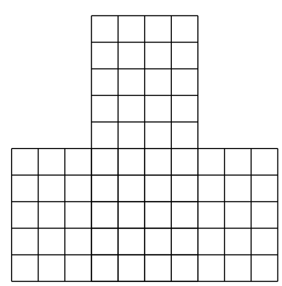
21. Consider the 9×9 grid of lattice points shown below. Points P and Q are given. How many points R in the grid are there for which triangle PQR is isosceles?



Next, let $P = (a, b)$ be a lattice point. Find necessary and sufficient conditions on P so that the set of points Q for which triangle PQO has integer area, where O is the origin, is finite.

22. How many subsets of $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ have either 5, 6 or 7 as their largest element?

23. How many rectangular regions are bounded by the gridlines of the figure below?



24. (Mathcounts 2010, Target Round) Seven identical red cards and three identical black cards are laid down in a row on a table. How many

distinguishable arrangements are possible if no two black cards are allowed to be adjacent to each other?

25. (old USAMO problem) In a math contest, three problems, A, B, and C were posed. Among the participants there were 25 who solved at least one problem. Of all the participants who did not solve problem A, the number who solved problem B was twice the number who solved C. The number who solved only problem A was one more than the number who solved A and at least one other problem. Of all participants who solved just one problem, half did not solve problem A. How many solved only problem B?

26. In a chess tournament, the number of boy participants is double the number of girl participants. Every two participants play exactly one game against each other. At the end of the tournament, no games were drawn. The ratio between the number of wins by the girls and the number of wins by the boys is 7:5. How many boys were there in the tournament?

27. Find the number of positive integer triples (x, y, z) satisfying $xy^2z^3 = 1,000,000$.

28. For how many different subsets of the set $S = \{2, 3, 4, 6, 15, 20, 30\}$ is the sum of the elements at least 50?