Integers and Rational Numbers

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CONCEPT 1 Integers and Rational Numbers

Learning Objectives

- Graph and compare integers.
- Classify and order rational numbers.
- Find opposites of numbers.
- Find absolute values.
- Compare fractions to determine which is bigger.

Introduction

One day, Jason leaves his house and starts walking to school. After three blocks, he stops to tie his shoe and leaves his lunch bag sitting on the curb. Two blocks farther on, he realizes his lunch is missing and goes back to get it. After picking up his lunch, he walks six more blocks to arrive at school. How far is the school from Jason's house? And how far did Jason actually walk to get there?

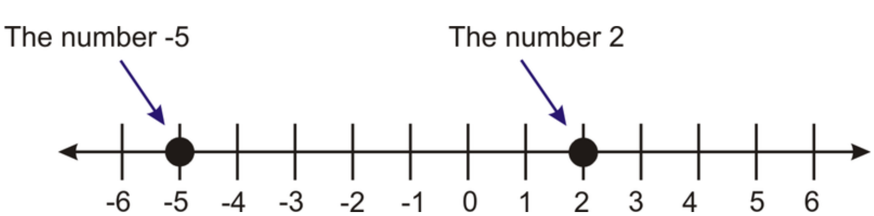
Graph and Compare Integers

Integers are the counting numbers (1, 2, 3...), the negative opposites of the counting numbers (-1, -2, -3...), and zero. There are an infinite number of integers and examples are 0, 3, 76, -2, -11, and 995.

Example 1

Compare the numbers 2 *and* -5.

When we plot numbers on a number line, the **greatest** number is farthest to the right, and the **least** is farthest to the left.



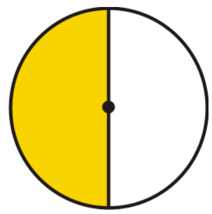
In the diagram above, we can see that 2 is farther to the right on the number line than -5, so we say that 2 is greater than -5. We use the symbol "$>$" to mean "greater than", so we can write $2 > 5$.

Classifying Rational Numbers

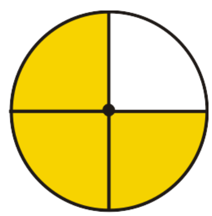
When we divide an integer a by another integer b (as long as b is not zero) we get a **rational number**. It's called this because it is the **ratio** of one number to another, and we can write it in fraction form as $\frac{a}{b}$. (You may recall that the top number in a fraction is called the **numerator** and the bottom number is called the **denominator**.)

You can think of a rational number as a fraction of a cake. If you cut the cake into b slices, your share is a of those slices.

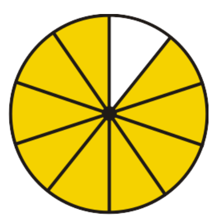
For example, when we see the rational number $\frac{1}{2}$, we can imagine cutting the cake into two parts. Our share is one of those parts. Visually, the rational number $\frac{1}{2}$ looks like this:



With the rational number $\frac{3}{4}$, we cut the cake into four parts and our share is three of those parts. Visually, the rational number $\frac{3}{4}$ looks like this:



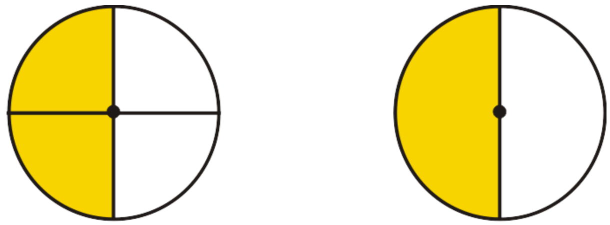
The rational number $\frac{9}{10}$ represents nine slices of a cake that has been cut into ten pieces. Visually, the rational number $\frac{9}{10}$ looks like this:



Proper fractions are rational numbers where the numerator is less than the denominator. A proper fraction represents a number less than one.

Improper fractions are rational numbers where the numerator is greater than or equal to the denominator. An improper fraction can be rewritten as a mixed number – an integer plus a proper fraction. For example, $\frac{9}{4}$ can be written as $2\frac{1}{4}$. An improper fraction represents a number greater than or equal to one.

Equivalent fractions are two fractions that represent the same amount. For example, look at a visual representation of the rational number $\frac{2}{4}$, and one of the number $\frac{1}{2}$.



You can see that the shaded regions are the same size, so the two fractions are equivalent. We can convert one fraction into the other by **reducing** the fraction, or writing it in lowest terms. To do this, we write out the prime factors of both the numerator and the denominator and cancel matching factors that appear in both the numerator **and** denominator.

$$\frac{2}{4} = \frac{2 \cdot 1}{2 \cdot 2 \cdot 1} = \frac{1}{2 \cdot 1} = \frac{1}{2}$$

Reducing a fraction doesn't change the value of the fraction—it just simplifies the way we write it. Once we've canceled all common factors, the fraction is in its **simplest form**.

Example 2

Classify and simplify the following rational numbers

a) $\frac{3}{7}$

b) $\frac{9}{3}$

c) $\frac{50}{60}$

Solution

a) 3 and 7 are both prime, so we can't factor them. That means $\frac{3}{7}$ is already in its simplest form. It is also a proper fraction.

b) $\frac{9}{3}$ is an improper fraction because $9 > 3$. To simplify it, we factor the numerator and denominator and cancel: $\frac{3 \cdot 3}{3 \cdot 1} = \frac{3}{1} = 3$.

c) $\frac{50}{60}$ is a proper fraction, and we can simplify it as follows: $\frac{50}{60} = \frac{5 \cdot 5 \cdot 2}{5 \cdot 3 \cdot 2 \cdot 2} = \frac{5}{3 \cdot 2} = \frac{5}{6}$.

Order Rational Numbers

Ordering rational numbers is simply a matter of arranging them by increasing value—least first and greatest last.

Example 3

Put the following fractions in order from least to greatest: $\frac{1}{2}, \frac{3}{4}, \frac{2}{3}$

Solution

$\frac{1}{2} < \frac{2}{3} < \frac{3}{4}$

Simple fractions are easy to order—we just know, for example, that one-half is greater than one quarter, and that two thirds is bigger than one-half. But how do we compare more complex fractions?

Example 4

Which is greater, $\frac{3}{7}$ *or* $\frac{4}{9}$*?*

In order to determine this, we need to rewrite the fractions so we can compare them more easily. If we rewrite them as equivalent fractions that have the same denominators, then we can compare them directly. To do this, we need to find the **lowest common denominator** (LCD), or the least common multiple of the two denominators.

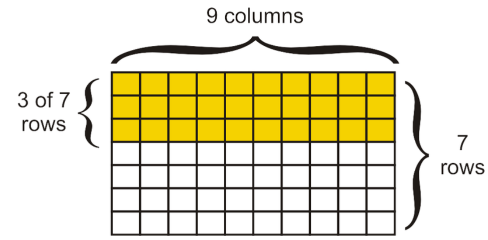
The lowest common multiple of 7 and 9 is 63. Our fraction will be represented by a shape divided into 63 sections. This time we will use a rectangle cut into 9 by $7 = 63$ pieces.

7 divides into 63 nine times, so $\frac{3}{7} = \frac{9 \cdot 3}{9 \cdot 7} = \frac{27}{63}$.

We can multiply the numerator and the denominator both by 9 because that's really just the opposite of reducing the fraction—to get back from $\frac{27}{63}$ to $\frac{3}{7}$, we'd just cancel out the 9's. Or, to put that in more formal terms:

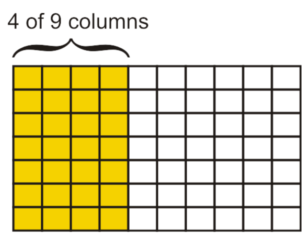
The fractions $\frac{a}{b}$ and $\frac{c \cdot a}{c \cdot b}$ are equivalent as long as $c \neq 0$.

Therefore, $\frac{27}{63}$ is an equivalent fraction to $\frac{3}{7}$. Here it is shown visually:

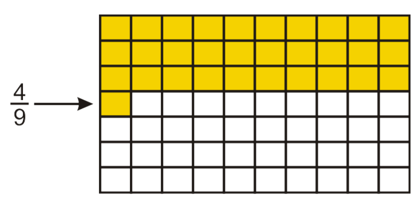


9 divides into 63 seven times, so $\frac{4}{9} = \frac{7 \cdot 4}{7 \cdot 9} = \frac{28}{63}$.

$\frac{28}{63}$ is an equivalent fraction to $\frac{4}{9}$. Here it is shown visually:



By writing the fractions with a **common denominator** of 63, we can easily compare them. If we take the 28 shaded boxes out of 63 (from our image of $\frac{4}{9}$ above) and arrange them in rows instead of columns, we can see that they take up more space than the 27 boxes from our image of $\frac{3}{7}$:



Solution

Since $\frac{28}{63}$ is greater than $\frac{27}{63}$, $\frac{4}{9}$ is greater than $\frac{3}{7}$.

Graph and Order Rational Numbers

To plot non-integer rational numbers (fractions) on the number line, we can convert them to mixed numbers (graphing is one of the few occasions in algebra when it's better to use mixed numbers than improper fractions), or we can convert them to decimal form.

Example 5

Plot the following rational numbers on the number line.

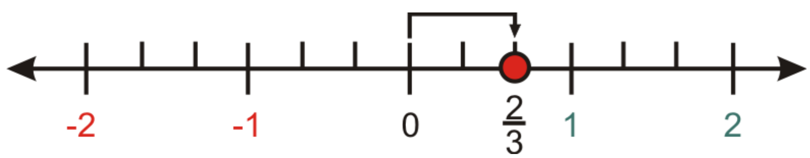
a) $\frac{2}{3}$

b) $-\frac{3}{7}$

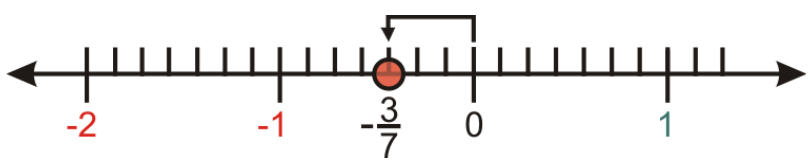
c) $\frac{17}{5}$

If we divide up the number line into sub-intervals based on the denominator of the fraction, we can look at the fraction's numerator to determine how many of these sub-intervals we need to include.

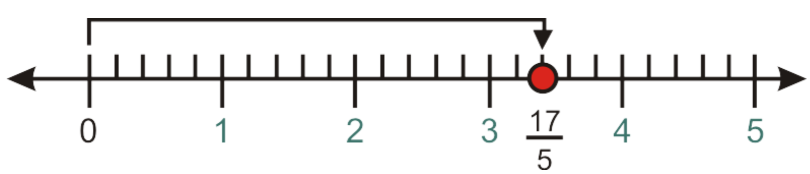
a) $\frac{2}{3}$ falls between 0 and 1. Because the denominator is 3, we divide the interval between 0 and 1 into three smaller units. Because the numerator is 2, we count two units over from 0.



b) $-\frac{3}{7}$ falls between 0 and -1. We divide the interval into seven units, and move left from zero by three of those units.



c) $\frac{17}{5}$ as a mixed number is $3\frac{2}{5}$ and falls between 3 and 4. We divide the interval into five units, and move over two units.

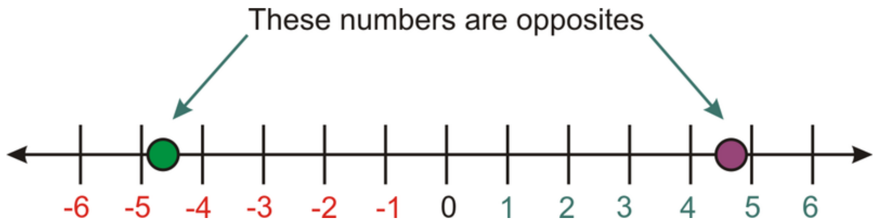


Another way to graph this fraction would be as a decimal. $3\frac{2}{5}$ is equal to 3.4, so instead of dividing the interval between 3 and 4 into 5 units, we could divide it into 10 units (each representing a distance of 0.1) and then count over 4 units. We would end up at the same place on the number line either way.

To make graphing rational numbers easier, try using the number line generator at http://themathworksheetsite.com/numline.html. You can use it to create a number line divided into whatever units you want, as long as you express the units in decimal form.

Find the Opposites of Numbers

Every number has an opposite. On the number line, a number and its opposite are, predictably, *opposite* each other. In other words, they are the same distance from zero, but on opposite sides of the number line.



The opposite of zero is defined to be simply zero.

The sum of a number and its opposite is always zero—for example, $3+-3=0, 4.2+-4.2=0$, and so on. This is because adding 3 and -3 is like moving 3 steps to the right along the number line, and then 3 steps back to the left. The number and its opposite cancel each other out, leaving zero.

Another way to think of the opposite of a number is that it is simply the original number multiplied by -1. The opposite of 4 is 4×-1 or -4, the opposite of -2.3 is -2.3×-1 or just 2.3, and so on. Another term for the opposite of a number is the **additive inverse**.

Example 6

Find the opposite of each of the following:

d) 19.6

e) $-\frac{4}{9}$

f) x

g) xy^2

h) $(x-3)$

Solution

Since we know that opposite numbers are on opposite sides of zero, we can simply multiply each expression by -1. This changes the sign of the number to its opposite—if it's negative, it becomes positive, and vice versa.

a) The opposite of 19.6 is -19.6.

b) The opposite of is $-\frac{4}{9}$ is $\frac{4}{9}$.

c) The opposite of x is $-x$.

d) The opposite of xy^2 is $-xy^2$.

e) The opposite of $(x-3)$ is $-(x-3)$, or $(3-x)$.

Note: With the last example you must multiply the **entire expression** by -1. A common mistake in this example is to assume that the opposite of $(x-3)$ is $(x+3)$. Avoid this mistake!

Find Absolute Values

When we talk about absolute value, we are talking about distances on the number line. For example, the number 7 is 7 units away from zero—and so is the number -7. The absolute value of a number is the distance it is from zero, so the absolute value of 7 and the absolute value of -7 are both 7.

We **write** the absolute value of -7 as $|-7|$. We **read** the expression $|x|$ as "the absolute value of x."

- Treat absolute value expressions like parentheses. If there is an operation inside the absolute value symbols, evaluate that operation first.

- The absolute value of a number or an expression is **always** positive or zero. It cannot be negative. With absolute value, we are only interested in how far a number is from zero, and not in which direction.

Example 7

Evaluate the following absolute value expressions.

a) $|5+4|$

b) $3-|4-9|$

c) $|-5-11|$

d) $-|7-22|$

(Remember to treat any expressions inside the absolute value sign as if they were inside parentheses, and evaluate them first.)

Solution

a) $|5+4| = |9| = 9$

b) $3-|4-9| = 3-|-5| = 3-5 = -2$

c) $|-5-11| = |-16| = 16$

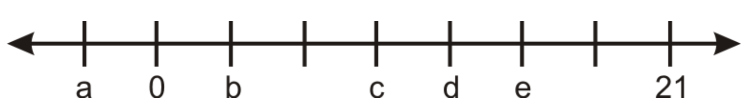
d) $-|7-22| = -|-15| = -(15) = -15$

Lesson Summary

- **Integers** (or **whole numbers**) are the counting numbers (1, 2, 3, ...), the negative counting numbers (-1, -2, -3, ...), and zero.
- A **rational number** is the **ratio** of one integer to another, like $\frac{3}{5}$ or $\frac{a}{b}$. The top number is called the **numerator** and the bottom number (which can't be zero) is called the **denominator**.
- **Proper fractions** are rational numbers where the numerator is less than the denominator.
- **Improper fractions** are rational numbers where the numerator is greater than the denominator.
- **Equivalent fractions** are two fractions that equal the same numerical value. The fractions $\frac{a}{b}$ and $\frac{c \cdot a}{c \cdot b}$ are equivalent as long as $c \neq 0$.
- To **reduce** a fraction (write it in **simplest form**), write out all prime factors of the numerator and denominator, cancel common factors, then recombine.
- To compare two fractions it helps to write them with a **common denominator**.
- The **absolute value** of a number is the distance it is from zero on the number line. The absolute value of any expression will always be positive or zero.
- Two numbers are **opposites** if they are the same distance from zero on the number line and on opposite sides of zero. The opposite of an expression can be found by multiplying **the entire expression** by -1.

Review Questions

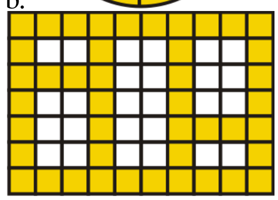
1. Solve the problem posed in the Introduction.
2. The tick-marks on the number line represent evenly spaced integers. Find the values of a, b, c, d and e.



3. Determine what fraction of the whole each shaded region represents.

a. 

b. 

c. 

4. Place the following sets of rational numbers in order, from least to greatest.

   a. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$
   b. $\frac{1}{10}, \frac{1}{2}, \frac{2}{5}, \frac{1}{4}, \frac{7}{20}$
   c. $\frac{39}{60}, \frac{49}{80}, \frac{59}{100}$
   d. $\frac{7}{11}, \frac{8}{13}, \frac{12}{19}$
   e. $\frac{9}{5}, \frac{22}{15}, \frac{4}{3}$

5. Find the simplest form of the following rational numbers.

   a. $\frac{22}{44}$
   b. $\frac{9}{27}$
   c. $\frac{12}{18}$
   d. $\frac{315}{420}$
   e. $\frac{244}{168}$

6. Find the opposite of each of the following.

   a. 1.001
   b. $(5-11)$
   c. $(x+y)$
   d. $(x-y)$
   e. $(x+y-4)$
   f. $(-x+2y)$

7. Simplify the following absolute value expressions.

   a. $11-|-4|$
   b. $|4-9|-|-5|$
   c. $|-5-11|$
   d. $7-|22-15-19|$
   e. $-|-7|$
   f. $|-2-88|-|88+2|$