Adding and Subtracting Rational Numbers

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CONCEPT 1 Adding and Subtracting Rational Numbers

Learning Objectives

- Add and subtract using a number line.
- Add and subtract rational numbers.
- Identify and apply properties of addition and subtraction.
- Solve real-world problems using addition and subtraction of fractions.
- Evaluate change using a variable expression.

Introduction

Ilana buys two identically sized cakes for a party. She cuts the chocolate cake into 24 pieces and the vanilla cake into 20 pieces, and lets the guests serve themselves. Martin takes three pieces of chocolate cake and one of vanilla, and Sheena takes one piece of chocolate and two of vanilla. Which of them gets more cake?

Add and Subtract Using a Number Line

In Lesson 1, we learned how to represent numbers on a number line. To add numbers on a number line, we start at the position of the first number, and then move to the right by a number of units equal to the second number.

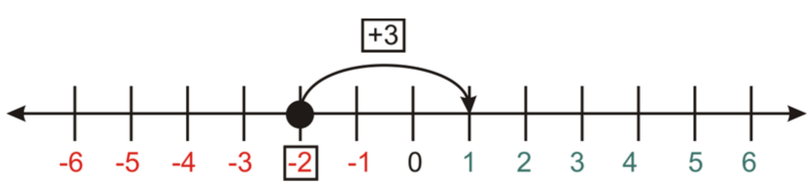
Example 1

Represent the sum $-2+3$ *on a number line.*

We start at the number -2, and then move 3 units to the right. We thus end at +1.

Solution

$-2+3=1$



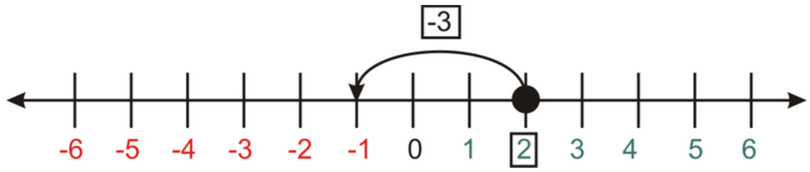
Example 2

Represent the sum 2 - 3 on a number line.

Subtracting a number is basically just **adding a negative number**. Instead of moving to the right, we move to the left. Starting at the number 2, and then moving 3 to the left, means we end at -1.

Solution

$2-3=-1$



Adding and Subtracting Rational Numbers

When we add or subtract two fractions, the denominators must match before we can find the sum or difference. We have already seen how to find a common denominator for two rational numbers.

Example 3

Simplify $\frac{3}{5}+\frac{1}{6}$.

To combine these fractions, we need to rewrite them over a common denominator. We are looking for the **lowest common denominator** (LCD). We need to identify the **lowest common multiple** or **least common multiple** (LCM) of 5 and 6. That is the smallest number that both 5 and 6 divide into evenly (that is, without a remainder).

The lowest number that 5 and 6 both divide into evenly is 30. The LCM of 5 and 6 is 30, so the lowest common denominator for our fractions is also 30.

We need to rewrite our fractions as new **equivalent fractions** so that the denominator in each case is 30.

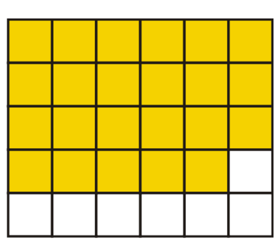
If you think back to our idea of a cake cut into a number of slices, $\frac{3}{5}$ means 3 slices of a cake that has been cut into 5 pieces. You can see that if we cut the same cake into 30 pieces (6 times as many) we would need 6 times as many slices to make up an equivalent fraction of the cake—in other words, 18 slices instead of 3.

$\frac{3}{5}$ is equivalent to $\frac{18}{30}$.

By a similar argument, we can rewrite the fraction $\frac{1}{6}$ as a share of a cake that has been cut into 30 pieces. If we cut it into 5 times as many pieces, we need 5 times as many slices.

$\frac{1}{6}$ is equivalent to $\frac{5}{30}$.

Now that both fractions have the same denominator, we can add them. If we add 18 pieces of cake to 5 pieces, we get a total of 23 pieces. 23 pieces of a cake that has been cut into 30 pieces means that our answer is $\frac{23}{30}$.



$$\frac{3}{5}+\frac{1}{6}=\frac{18}{30}+\frac{5}{30}=\frac{23}{30}$$

Notice that when we have fractions with a common denominator, we **add the numerators** but we **leave the denominators alone**. Here is this information in algebraic terms.

When adding fractions: $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$

Example 4

Simplify $\frac{1}{3}-\frac{1}{9}$.

The lowest common multiple of 9 and 3 is 9, so 9 is our common denominator. That means we don't have to alter the second fraction at all.

3 divides into 9 three times, so $\frac{1}{3}=\frac{3\cdot 1}{3\cdot 3}=\frac{3}{9}$. Our sum becomes $\frac{3}{9}-\frac{1}{9}$. We can subtract fractions with a common denominator by subtracting their numerators, just like adding. In other words:

When subtracting fractions: $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$

Solution

$\frac{1}{3}-\frac{1}{9}=\frac{2}{9}$

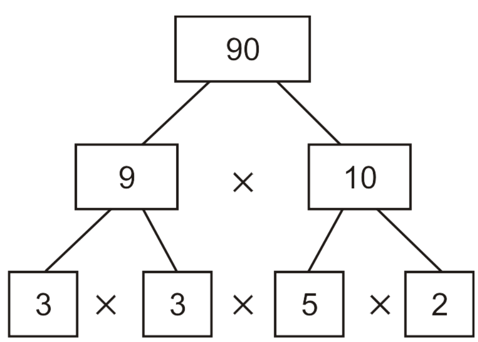
So far, we've only dealt with examples where it's easy to find the least common multiple of the denominators. With larger numbers, it isn't so easy to be sure that we have the LCD. We need a more systematic method. In the next example, we will use the method of **prime factors** to find the least common denominator.

Example 5

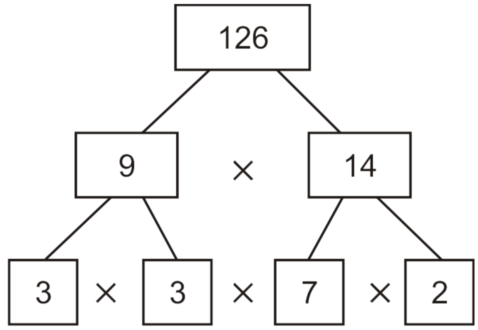
Simplify $\frac{29}{90}-\frac{13}{126}$.

To find the lowest common multiple of 90 and 126, we first find the prime factors of 90 and 126. We do this by continually dividing the number by factors until we can't divide any further. You may have seen a factor tree before. (For practice creating factor trees, try the Factor Tree game at http://www.mathgoodies.com/factors/factor_tree.asp.)

The factor tree for 90 looks like this:



The factor tree for 126 looks like this:



The LCM for 90 and 126 is made from the **smallest possible collection of primes** that enables us to construct either of the two numbers. We take only enough instances of each prime to make the number with the greater number of instances of that prime in its factor tree.

TABLE 1.1:

Prime	**Factors in 90**	**Factors in 126**	**We Need**
2	1	1	1
3	2	2	2
5	1	0	1
7	0	1	1

So we need one 2, two 3's, one 5 and one 7. That gives us $2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 = 630$ as the lowest common multiple of 90 and 126. So 630 is the LCD for our calculation.

90 divides into 630 seven times (notice that 7 is the only factor in 630 that is missing from 90), so $\frac{29}{90} = \frac{7 \cdot 29}{7 \cdot 90} = \frac{203}{630}$.

126 divides into 630 five times (notice that 5 is the only factor in 630 that is missing from 126), so $\frac{13}{126} = \frac{5 \cdot 13}{5 \cdot 126} = \frac{65}{630}$.

Now we complete the problem: $\frac{29}{90} - \frac{13}{126} = \frac{203}{630} - \frac{65}{630} = \frac{138}{630}$.

This fraction **simplifies**. To be sure of finding the **simplest form** for $\frac{138}{630}$, we write out the prime factors of the numerator and denominator. We already know the prime factors of 630. The prime factors of 138 are 2, 3 and 23.

$\frac{138}{630} = \frac{2 \cdot 3 \cdot 23}{2 \cdot 3 \cdot 3 \cdot 5 \cdot 7}$; one factor of 2 and one factor of 3 cancels out, leaving $\frac{23}{3 \cdot 5 \cdot 7}$ or $\frac{23}{105}$ as our answer.

Identify and Apply Properties of Addition

Three mathematical properties which involve addition are the **commutative, associative,** and the **additive identity properties**.

Commutative property: When two numbers are added, the sum is the same even if the order of the items being added changes.

Example: $3+2=2+3$

Associative Property: When three or more numbers are added, the sum is the same regardless of how they are grouped.

Example: $(2+3)+4=2+(3+4)$

Additive Identity Property: The sum of any number and zero is the original number.

Example: $5+0=5$

Solve Real-World Problems Using Addition and Subtraction

Example 6

Peter is hoping to travel on a school trip to Europe. The ticket costs $2400. Peter has several relatives who have pledged to help him with the ticket cost. His parents have told him that they will cover half the cost. His grandma Zenoviea will pay one sixth, and his grandparents in Florida will send him one fourth of the cost. What fraction of the cost can Peter count on his relatives to provide?

The first thing we need to do is extract the relevant information. Peter's parents will provide $\frac{1}{2}$ the cost; his grandma Zenoviea will provide $\frac{1}{6}$; and his grandparents in Florida $\frac{1}{4}$. We need to find the sum of those numbers, or $\frac{1}{2}+\frac{1}{6}+\frac{1}{4}$.

To determine the sum, we first need to find the LCD. The LCM of 2, 6 and 4 is 12, so that's our LCD. Now we can find equivalent fractions:

$$\frac{1}{2}=\frac{6\cdot 1}{6\cdot 2}=\frac{6}{12}$$
$$\frac{1}{6}=\frac{2\cdot 1}{2\cdot 6}=\frac{2}{12}$$
$$\frac{1}{4}=\frac{3\cdot 1}{3\cdot 4}=\frac{3}{12}$$

Putting them all together: $\frac{6}{12}+\frac{2}{12}+\frac{3}{12}=\frac{11}{12}$.

Peter will get $\frac{11}{12}$ the cost of the trip, or $2200 out of $2400, from his family.

Example 7

A property management firm is buying parcels of land in order to build a small community of condominiums. It has just bought three adjacent plots of land. The first is four-fifths of an acre, the second is five-twelfths of an acre, and the third is nineteen-twentieths of an acre. The firm knows that it must allow one-sixth of an acre for utilities and a small access road. How much of the remaining land is available for development?

The first thing we need to do is extract the relevant information. The plots of land measure $\frac{4}{5}$, $\frac{5}{12}$, and $\frac{19}{20}$ acres, and the firm can use all of that land except for $\frac{1}{6}$ of an acre. The total amount of land the firm can use is therefore $\frac{4}{5}+\frac{5}{12}+\frac{19}{20}-\frac{1}{6}$ acres.

We can add and subtract multiple fractions at once just by finding a common denominator for all of them. The factors of 5, 9, 20, and 6 are as follows:

5	5
12	$2\cdot 2\cdot 3$
20	$2\cdot 2\cdot 5$
6	$2\cdot 3$

We need a 5, two 2's, and a 3 in our LCD. $2\cdot 2\cdot 3\cdot 5=60$, so that's our common denominator. Now to convert the fractions:

$$\frac{4}{5}=\frac{12\cdot4}{12\cdot5}=\frac{48}{60}$$
$$\frac{5}{12}=\frac{5\cdot5}{5\cdot12}=\frac{25}{60}$$
$$\frac{19}{20}=\frac{3\cdot19}{3\cdot20}=\frac{57}{60}$$
$$\frac{1}{6}=\frac{10\cdot1}{10\cdot6}=\frac{10}{60}$$

We can rewrite our sum as $\frac{48}{60}+\frac{25}{60}+\frac{57}{60}-\frac{10}{60}=\frac{48+25+57-10}{60}=\frac{120}{60}$.

Next, we need to reduce this fraction. We can see immediately that the numerator is twice the denominator, so this fraction reduces to $\frac{2}{1}$ or simply 2. One is sometimes called the **invisible denominator**, because every whole number can be thought of as a rational number whose denominator is one.

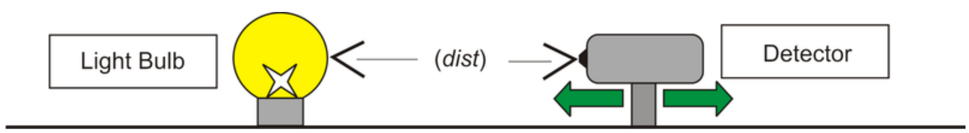
Solution

The property firm has two acres available for development.

Evaluate Change Using a Variable Expression

When we write algebraic expressions to represent a real quantity, the difference between two values is the **change** in that quantity.

Example 8



The intensity of light hitting a detector when it is held a certain distance from a bulb is given by this equation:

$$Intensity=\frac{3}{d^2}$$

*where d is the distance measured in **meters**, and intensity is measured in **lumens**. Calculate the change in intensity when the detector is moved from two meters to three meters away.*

We first find the values of the intensity at distances of two and three meters.

$$\text{Intensity }(2)=\frac{3}{(2)^2}=\frac{3}{4}$$
$$\text{Intensity }(3)=\frac{3}{(3)^2}=\frac{3}{9}=\frac{1}{3}$$

The **difference** in the two values will give the **change** in the intensity. We move **from** two meters **to** three meters away.

Change $=$ Intensity $(3)-$ Intensity $(2)=\frac{1}{3}-\frac{3}{4}$

To find the answer, we will need to write these fractions over a common denominator.

The LCM of 3 and 4 is 12, so we need to rewrite each fraction with a denominator of 12:

$$\frac{1}{3} = \frac{4 \cdot 1}{4 \cdot 3} = \frac{4}{12}$$
$$\frac{3}{4} = \frac{3 \cdot 3}{3 \cdot 4} = \frac{9}{12}$$

So we can rewrite our equation as $\frac{4}{12} - \frac{9}{12} = -\frac{5}{12}$. The negative value means that the intensity decreases as we move from 2 to 3 meters away.

Solution

When moving the detector from two meters to three meters, the intensity falls by $\frac{5}{12}$ lumens.

Lesson Summary

- **Subtracting** a number is the same as adding the **opposite** (or **additive inverse**) of the number.
- To add fractions, rewrite them over the **lowest common denominator (LCD)**. The lowest common denominator is the **lowest** (or **least**) **common multiple (LCM)** of the two denominators.
- When **adding fractions:** $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$
- When **subtracting fractions:** $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$
- **Commutative property:** the sum of two numbers is the same even if the order of the items to be added changes.
- **Associative Property:** When three or more numbers are added, the sum is the same regardless of how they are grouped.
- **Additive Identity Property:** The sum of any number and zero is the original number.
- The number one is sometimes called the **invisible denominator**, as every whole number can be thought of as a rational number whose denominator is one.
- The **difference** between two values is the **change** in that quantity.

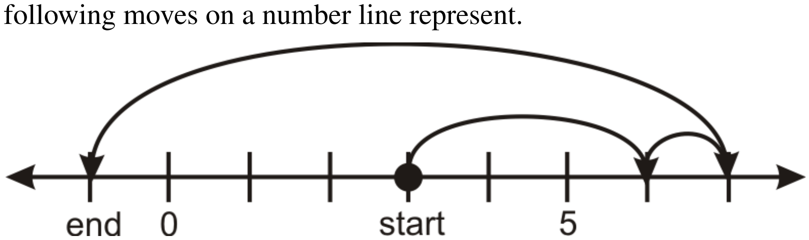
Further Practice

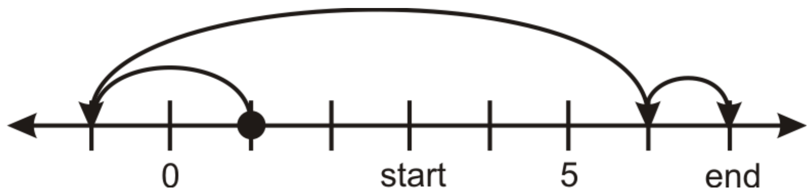
For more practice adding and subtracting fractions, try playing the math games at http://www.mathplayground.com/fractions_add.html and http://www.mathplayground.com/fractions_sub.html, or the one at http://www.aaamath.com/fra66kx2.htm.

Review Questions

1. Write the sum that the following moves on a number line represent.

a.



b. 

2. Add the following rational numbers. Write each answer in its **simplest form**.
   a. $\frac{3}{7}+\frac{2}{7}$
   b. $\frac{3}{10}+\frac{1}{5}$
   c. $\frac{5}{16}+\frac{5}{12}$
   d. $\frac{3}{8}+\frac{9}{16}$
   e. $\frac{8}{25}+\frac{7}{10}$
   f. $\frac{1}{6}+\frac{1}{4}$
   g. $\frac{7}{15}+\frac{2}{9}$
   h. $\frac{5}{19}+\frac{2}{27}$
3. Which property of addition does each situation involve?
   a. Whichever order your groceries are scanned at the store, the total will be the same.
   b. However many shovel-loads it takes to move 1 ton of gravel, the number of rocks moved is the same.
   c. If Julia has no money, then Mark and Julia together have just as much money as Mark by himself has.
4. Solve the problem posed in the Introduction to this lesson.
5. Nadia, Peter and Ian are pooling their money to buy a gallon of ice cream. Nadia is the oldest and gets the greatest allowance. She contributes half of the cost. Ian is next oldest and contributes one third of the cost. Peter, the youngest, gets the smallest allowance and contributes one fourth of the cost. They figure that this will be enough money. When they get to the check-out, they realize that they forgot about sales tax and worry there will not be enough money. Amazingly, they have exactly the right amount of money. What fraction of the cost of the ice cream was added as tax?
6. Subtract the following rational numbers. Be sure that your answer is in the **simplest form**.
   a. $\frac{5}{12}-\frac{9}{18}$
   b. $\frac{2}{3}-\frac{1}{4}$
   c. $\frac{3}{4}-\frac{1}{3}$
   d. $\frac{15}{11}-\frac{9}{7}$
   e. $\frac{2}{13}-\frac{1}{11}$
   f. $\frac{7}{27}-\frac{9}{39}$
   g. $\frac{6}{11}-\frac{3}{22}$
   h. $\frac{13}{64}-\frac{7}{40}$
   i. $\frac{11}{70}-\frac{11}{30}$
7. Consider the equation $y=3x+2$. Determine the change in y between $x=3$ and $x=7$.
8. Consider the equation $y=\frac{2}{3}x+\frac{1}{2}$. Determine the change in y between $x=1$ and $x=2$.
9. The time taken to commute from San Diego to Los Angeles is given by the equation $time=\frac{120}{speed}$ where *time* is measured in **hours** and *speed* is measured in **miles per hour** (mph). Calculate the change in time that a rush hour commuter would see when switching from traveling by bus to traveling by train, if the bus averages 40 mph and the train averages 90 mph.