Multiplying and Dividing Rational Numbers

CK12 Editor

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CONCEPT 1

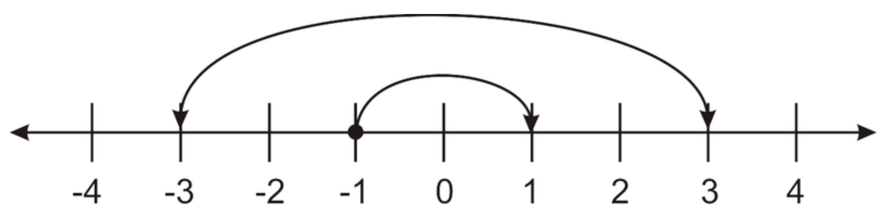
Multiplying and Dividing Rational Numbers

Learning Objectives

- Multiply by negative one.
- Multiply rational numbers.
- Identify and apply properties of multiplication.
- Solve real-world problems using multiplication.
- Find multiplicative inverses.
- Divide rational numbers.
- Solve real-world problems using division.

Multiplying Numbers by Negative One

Whenever we multiply a number by negative one, the sign of the number changes. In more mathematical terms, multiplying by negative one maps a number onto its opposite. The number line below shows two examples: $3 \cdot -1 = 3$ and $-1 \cdot -1 = 1$.



When we multiply a number by negative one, the absolute value of the new number is the same as the absolute value of the old number, since both numbers are the same distance from zero.

The product of a number "x" and negative one is $-x$. This does not mean that $-x$ is necessarily less than zero! If x itself is negative, then $-x$ will be positive because a negative times a negative (negative one) is a positive.

When you multiply an expression by negative one, remember to multiply the **entire expression** by negative one.

Example 1

Multiply the following by negative one.

a) 79.5

b) π

c) $(x+1)$

d) $|x|$

Solution

a) -79.5

b) $-\pi$

c) $-(x+1)$ or $-x-1$

d) $-|x|$

Note that in the last case the negative sign **outside** the absolute value symbol applies **after** the absolute value. Multiplying the **argument** of an absolute value equation (the term inside the absolute value symbol) does not change the absolute value. $|x|$ is always positive. $|-x|$ is always positive. $-|x|$ is always negative.

Whenever you are working with expressions, you can check your answers by substituting in numbers for the variables. For example, you could check part *d* of Example 1 by letting $x = -3$. Then you'd see that $|-3| \neq -|3|$, because $|-3| = 3$ and $-|3| = -3$.

Careful, though—plugging in numbers can tell you if your answer is wrong, but it won't always tell you for sure if your answer is right!

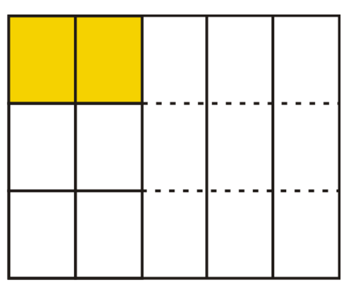
Multiply Rational Numbers

Example 2

Simplify $\frac{1}{3} \cdot \frac{2}{5}$.

One way to solve this is to think of money. For example, we know that *one third of sixty dollars* is written as $\frac{1}{3} \cdot \$60$. We can read the above problem as *one-third of two-fifths*. Here is a visual picture of the fractions ***one-third*** and ***two-fifths***.

If we divide our rectangle into thirds one way and fifths the other way, here's what we get:



Here is the intersection of the two shaded regions. The whole has been divided into five pieces width-wise and three pieces height-wise. We get two pieces out of a total of fifteen pieces.

Solution

$\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}$

Notice that $1 \cdot 2 = 2$ and $3 \cdot 5 = 15$. This turns out to be true in general: when you multiply rational numbers, the numerators multiply together and the denominators multiply together. Or, to put it more formally:

When multiplying fractions: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

This rule doesn't just hold for the product of two fractions, but for any number of fractions.

Example 4

Multiply the following rational numbers:

a) $\frac{2}{5} \cdot \frac{5}{9}$

b) $\frac{1}{3} \cdot \frac{2}{7} \cdot \frac{2}{5}$

c) $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5}$

Solution

a) With this problem, we can cancel the fives: $\frac{2}{5} \cdot \frac{5}{9} = \frac{2 \cdot 5}{5 \cdot 9} = \frac{2}{9}$.

b) With this problem, we multiply **all the numerators** and **all the denominators**:

$$\frac{1}{3} \cdot \frac{2}{7} \cdot \frac{2}{5} = \frac{1 \cdot 2 \cdot 2}{3 \cdot 7 \cdot 5} = \frac{4}{105}$$

c) With this problem, we multiply all the numerators and all the denominators, and then we can cancel most of them. The 2's, 3's, and 4's all cancel out, leaving $\frac{1}{5}$.

With multiplication of fractions, we can simplify before or after we multiply. The next example uses factors to help simplify before we multiply.

Example 5

Evaluate and simplify $\frac{12}{25} \cdot \frac{35}{42}$.

Solution

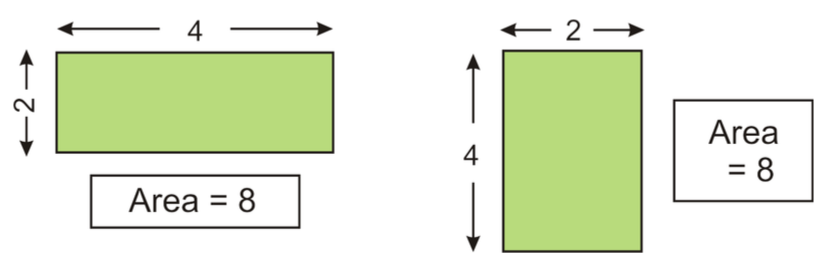
We can see that 12 and 42 are both multiples of six, 25 and 35 are both multiples of five, and 35 and 42 are both multiples of 7. That means we can write the whole product as $\frac{6 \cdot 2}{5 \cdot 5} \cdot \frac{5 \cdot 7}{6 \cdot 7} = \frac{6 \cdot 2 \cdot 5 \cdot 7}{5 \cdot 5 \cdot 6 \cdot 7}$. Then we can cancel out the 5, the 6, and the 7, leaving $\frac{2}{5}$.

Identify and Apply Properties of Multiplication

The four mathematical properties which involve multiplication are the **Commutative, Associative, Multiplicative Identity** and **Distributive Properties**.

Commutative property: When two numbers are multiplied together, the product is the same regardless of the order in which they are written.

Example: $4 \cdot 2 = 2 \cdot 4$



We can see a geometrical interpretation of **The Commutative Property of Multiplication** *to the right. The Area of the shape* $(length \times width)$ *is the same no matter which way we draw it.*

Associative Property: When three or more numbers are multiplied, the product is the same regardless of their grouping.

Example: $2 \cdot (3 \cdot 4) = (2 \cdot 3) \cdot 4$

Multiplicative Identity Property: The product of one and any number is that number.

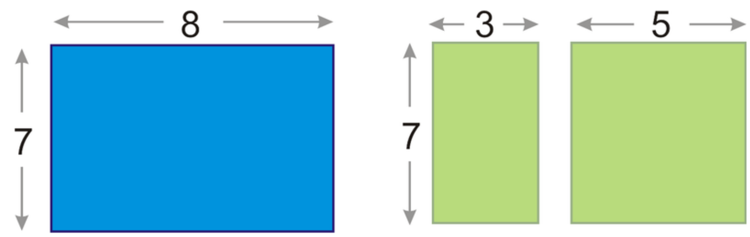
Example: $5 \cdot 1 = 5$

Distributive property: The multiplication of a number and the sum of two numbers is equal to the first number times the second number plus the first number times the third number.

Example: $4(6+3) = 4 \cdot 6 + 4 \cdot 3$

Example 6

A gardener is planting vegetables for the coming growing season. He wishes to plant potatoes and has a choice of a single 8×7 meter plot, or two smaller plots of 3×7 and 5×7 meters. Which option gives him the largest area for his potatoes?



Solution

In the first option, the gardener has a total area of (8×7) or 56 square meters.

In the second option, the gardener has (3×7) or 21 square meters, plus (5×7) or 35 square meters. $21 + 35 = 56$, so the area is the same as in the first option.

Solve Real-World Problems Using Multiplication

Example 7

In the chemistry lab there is a bottle with two liters of a 15% solution of hydrogen peroxide (H_2O_2). John removes one-fifth of what is in the bottle, and puts it in a beaker. He measures the amount of H_2O_2 and adds twice that amount of water to the beaker. Calculate the following measurements.

a) The amount of H_2O_2 left in the bottle.

b) The amount of diluted H_2O_2 in the beaker.

c) The concentration of the H_2O_2 in the beaker.

Solution

a) To determine the amount of H_2O_2 left in the bottle, we first determine the amount that was removed. That amount was $\frac{1}{5}$ of the amount in the bottle (2 liters). $\frac{1}{5}$ of 2 is $\frac{2}{5}$.

The amount remaining is $2 - \frac{2}{5}$, or $\frac{10}{5} - \frac{2}{5} = \frac{8}{5}$ liter (or 1.6 liters).

There are 1.6 liters left in the bottle.

b) We determined that the amount of the 15% H_2O_2 solution removed was $\frac{2}{5}$ liter. The amount of water added was twice this amount, or $\frac{4}{5}$ liter. So the total amount of solution in the beaker is now $\frac{2}{5} + \frac{4}{5} = \frac{6}{5}$ liter, or 1.2 liters.

There are 1.2 liters of diluted H_2O_2 in the beaker.

c) The new concentration of H_2O_2 can be calculated.

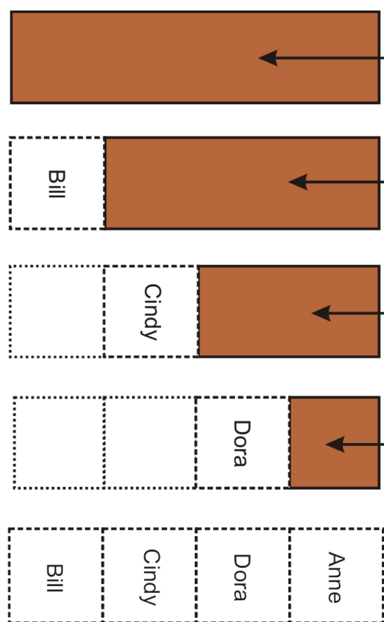
John started with $\frac{2}{5}$ liter of 15% H_2O_2 solution, so the amount of **pure** H_2O_2 is 15% of $\frac{2}{5}$ liters, or $0.15 \times 0.40 = 0.06$ liters.

After he adds the water, there is 1.2 liters of solution in the beaker, so the concentration of H_2O_2 is $\frac{0.06}{1.2} = \frac{1}{20}$ or 0.05. To convert to a percent we multiply this number by 100, so the beaker's contents are 5% H_2O_2.

Example 8

Anne has a bar of chocolate and she offers Bill a piece. Bill quickly breaks off $\frac{1}{4}$ of the bar and eats it. Another friend, Cindy, takes $\frac{1}{3}$ of what was left. Anne splits the remaining candy bar into two equal pieces which she shares with a third friend, Dora. How much of the candy bar does each person get?

First, let's look at this problem visually.



Anne starts with a full candy bar.

Bill breaks off $\frac{1}{4}$ of the bar.

Cindy takes $\frac{1}{3}$ of what was left.

Dora gets half of the remaining candy bar.

We can see that the candy bar ends up being split four ways, with each person getting an equal amount.

Solution

Each person gets exactly $\frac{1}{4}$ of the candy bar.

We can also examine this problem using rational numbers. We keep a running total of what fraction of the bar remains. Remember, when we read a fraction followed by *of* in the problem, it means we multiply by that fraction.

We start with 1 bar. Then Bill takes $\frac{1}{4}$ of it, so there is $1 - \frac{1}{4} = \frac{3}{4}$ of a bar left.

Cindy takes $\frac{1}{3}$ of what's left, or $\frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$ of a whole bar. That leaves $\frac{3}{4} - \frac{1}{4} = \frac{2}{4}$, or $\frac{1}{2}$ of a bar.

That half bar gets split between Anne and Dora, so they each get half of a half bar: $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

So each person gets exactly $\frac{1}{4}$ of the candy bar.

Extension: If each person's share is 3 oz, how much did the original candy bar weigh?

Identity Elements

An **identity element** is a number which, when combined with a mathematical operation on a number, leaves that number unchanged. For example, the **identity element** for addition and subtraction is **zero**, because adding or subtracting zero to a number doesn't change the number. And zero is also what you get when you add together a number and its opposite, like 3 and -3.

The **inverse operation** of addition is subtraction—when you add a number and then subtract that same number, you end up back where you started. Also, adding a number's opposite is the same as subtracting it—for example, $4+(-3)$ is the same as $4-3$.

Multiplication and division are also inverse operations to each other—when you multiply by a number and then divide by the same number, you end up back where you started. Multiplication and division also have an identity element: when you multiply or divide a number by **one**, the number doesn't change.

Just as the **opposite** of a number is the number you can add to it to get zero, the **reciprocal** of a number is the number you can multiply it by to get one. And finally, just as adding a number's opposite is the same as subtracting the number, multiplying by a number's reciprocal is the same as dividing by the number.

Find Multiplicative Inverses

The reciprocal of a number x is also called the **multiplicative inverse**. Any number times its own multiplicative inverse equals one, and the multiplicative inverse of x is written as $\frac{1}{x}$.

To find the multiplicative inverse of a rational number, we simply ***invert the fraction***—that is, flip it over. In other words:

The multiplicative inverse of $\frac{a}{b}$ is $\frac{b}{a}$, as long as $a \neq 0$.

You'll see why in the following exercise.

Example 9

Find the multiplicative inverse of each of the following.

a) $\frac{3}{7}$

b) $\frac{4}{9}$

c) $3\frac{1}{2}$

d) $-\frac{x}{y}$

e) $\frac{1}{11}$

Solution

a) When we invert the fraction $\frac{3}{7}$, we get $\frac{7}{3}$. Notice that if we multiply $\frac{3}{7} \cdot \frac{7}{3}$, the 3's and the 7's both cancel out and we end up with $\frac{1}{1}$, or just 1.

b) Similarly, the inverse of $\frac{4}{9}$ is $\frac{9}{4}$; if we multiply those two fractions together, the 4's and the 9's cancel out and we're left with 1. That's why the rule "invert the fraction to find the multiplicative inverse" works: the numerator and the denominator always end up canceling out, leaving 1.

c) To find the multiplicative inverse of $3\frac{1}{2}$ we first need to convert it to an improper fraction. Three wholes is six halves, so $3\frac{1}{2} = \frac{6}{2} + \frac{1}{2} = \frac{7}{2}$. That means the inverse is $\frac{2}{7}$.

d) Don't let the negative sign confuse you. The multiplicative inverse of a negative number is also negative! Just ignore the negative sign and flip the fraction as usual.

The multiplicative inverse of $-\frac{x}{y}$ is $-\frac{y}{x}$.

e) The multiplicative inverse of $\frac{1}{11}$ is $\frac{11}{1}$, or simply 11.

Look again at the last example. When we took the multiplicative inverse of $\frac{1}{11}$ we got a whole number, 11. That's because we can treat that whole number like a fraction with a denominator of 1. Any number, even a non-rational one, can be treated this way, so we can always find a number's multiplicative inverse using the same method.

Divide Rational Numbers

Earlier, we mentioned that multiplying by a number's reciprocal is the same as dividing by the number. That's how we can divide rational numbers; to divide by a rational number, just multiply by that number's reciprocal. In more formal terms:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

Example 10

Divide the following rational numbers, giving your answer in the **simplest form**.

a) $\frac{1}{2} \div \frac{1}{4}$

b) $\frac{7}{3} \div \frac{2}{3}$

c) $\frac{x}{2} \div \frac{1}{4y}$

d) $\frac{11}{2x} \div \left(-\frac{x}{y}\right)$

Solution

a) Replace $\frac{1}{4}$ with $\frac{4}{1}$ and multiply: $\frac{1}{2} \times \frac{4}{1} = \frac{4}{2} = 2$.

b) Replace $\frac{2}{3}$ with $\frac{3}{2}$ and multiply: $\frac{7}{3} \times \frac{3}{2} = \frac{7 \cdot 3}{3 \cdot 2} = \frac{7}{2}$.

c) $\frac{x}{2} \div \frac{1}{4y} = \frac{x}{2} \times \frac{4y}{1} = \frac{4xy}{2} = \frac{2xy}{1} = 2xy$

d) $\frac{11}{2x} \div \left(-\frac{x}{y}\right) = \frac{11}{2x} \times \left(-\frac{y}{x}\right) = -\frac{11y}{2x^2}$

Solve Real-World Problems Using Division

Speed, Distance and Time

An object moving at a certain **speed** will cover a fixed **distance** in a set **time**. The quantities speed, distance and time are related through the equation $\text{Speed} = \frac{\text{Distance}}{\text{Time}}$.

Example 11

Andrew is driving down the freeway. He passes mile marker 27 at exactly mid-day. At 12:35 he passes mile marker 69. At what speed, in miles per hour, is Andrew traveling?

Solution

To find the speed, we need the distance traveled and the time taken. If we want our speed to come out in miles per hour, we'll need distance in **miles** and time in **hours**.

The distance is $69-27$ or 42 miles. The time is 35 minutes, or $\frac{35}{60}$ hours, which reduces to $\frac{7}{12}$. Now we can *plug in* the values for distance and time into our equation for speed.

$$\text{Speed} = \frac{42}{\frac{7}{12}} = 42 \div \frac{7}{12} = \frac{42}{1} \times \frac{12}{7} = \frac{6 \cdot 7 \cdot 12}{1 \cdot 7} = \frac{6 \cdot 12}{1} = 72$$

Andrew is driving at 72 miles per hour.

Example 12

Anne runs a mile and a half in a quarter hour. What is her speed in miles per hour?

Solution

We already have the distance and time in the correct units (miles and hours), so we just need to write them as fractions and plug them into the equation.

$$\text{Speed} = \frac{1\frac{1}{2}}{\frac{1}{4}} = \frac{3}{2} \div \frac{1}{4} = \frac{3}{2} \times \frac{4}{1} = \frac{3 \cdot 4}{2 \cdot 1} = \frac{12}{2} = 6$$

Anne runs at 6 miles per hour.

Example 13 – Newton's Second Law

Newton's second law $(F = ma)$*relates the force applied to a body in Newtons* (F), *the mass of the body in kilograms* (m)*and the acceleration in meters per second squared* (a). *Calculate the resulting acceleration if a Force of* $7\frac{1}{3}$*Newtons is applied to a mass of* $\frac{1}{5}kg$.

Solution

First, we rearrange our equation to isolate the acceleration, a. If $F = ma$, dividing both sides by m gives us $a = \frac{F}{m}$. Then we substitute in the known values for F and m:

$$a = \frac{7\frac{1}{3}}{\frac{1}{5}} = \frac{22}{3} \div \frac{1}{5} = \frac{22}{3} \times \frac{5}{1} = \frac{110}{3}$$

The resultant acceleration is $36\frac{2}{3}\ m/s^2$.

Lesson Summary

When multiplying an expression by negative one, remember to multiply the **entire expression** by negative one.

To multiply fractions, multiply the numerators and multiply the denominators: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

The **multiplicative properties** are:

- **Commutative Property:** The product of two numbers is the same whichever order the items to be multiplied are written. **Example:** $2 \cdot 3 = 3 \cdot 2$
- **Associative Property:** When three or more numbers are multiplied, the sum is the same regardless of how they are grouped. **Example:** $2 \cdot (3 \cdot 4) = (2 \cdot 3) \cdot 4$

- **Multiplicative Identity Property:** The product of any number and one is the original number. **Example:** $2 \cdot 1 = 2$
- **Distributive property:** The multiplication of a number and the sum of two numbers is equal to the first number times the second number plus the first number times the third number. **Example:** $4(2+3) = 4(2)+4(3)$

The **multiplicative inverse** of a number is the number which produces 1 when multiplied by the original number. The multiplicative inverse of x is the reciprocal $\frac{1}{x}$. To find the multiplicative inverse of a fraction, simply **invert the fraction**: $\frac{a}{b}$ inverts to $\frac{b}{a}$.

To divide fractions, invert the divisor and multiply: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

Further Practice

For more practice multiplying fractions, try playing the fraction game at http://www.aaamath.com/fra66mx2.htm, or the one at http://www.mathplayground.com/fractions_mult.html. For more practice dividing fractions, try the game at http://www.aaamath.com/div66ox2.htm or the one at http://www.mathplayground.com/fractions_div.html.

Review Questions

1. Multiply the following expressions by negative one.
   a. 25
   b. -105
   c. x^2
   d. $(3+x)$
   e. $(3-x)$
2. Multiply the following rational numbers. Write your answer in the **simplest form**.
   a. $\frac{5}{12} \times \frac{9}{10}$
   b. $\frac{2}{3} \times \frac{1}{4}$
   c. $\frac{3}{4} \times \frac{1}{3}$
   d. $\frac{15}{11} \times \frac{9}{7}$
   e. $\frac{1}{13} \times \frac{1}{11}$
   f. $\frac{7}{27} \times \frac{9}{14}$
   g. $\left(\frac{3}{5}\right)^2$
   h. $\frac{1}{11} \times \frac{22}{21} \times \frac{7}{10}$
   i. $\frac{12}{15} \times \frac{35}{13} \times \frac{10}{2} \times \frac{26}{36}$
3. Find the multiplicative inverse of each of the following.
   a. 100
   b. $\frac{2}{8}$
   c. $-\frac{19}{21}$
   d. 7
   e. $-\frac{z^3}{2xy^2}$
4. Divide the following rational numbers. Write your answer in the simplest form.
   a. $\frac{5}{2} \div \frac{1}{4}$
   b. $\frac{1}{2} \div \frac{7}{9}$

c. $\frac{5}{11} \div \frac{6}{7}$
d. $\frac{1}{2} \div \frac{1}{2}$
e. $-\frac{x}{2} \div \frac{5}{7}$
f. $\frac{1}{2} \div \frac{x}{4y}$
g. $\left(-\frac{1}{3}\right) \div \left(-\frac{3}{5}\right)$
h. $\frac{7}{2} \div \frac{7}{4}$
i. $11 \div \frac{-x}{4}$

5. The label on a can of paint says that it will cover 50 square feet per pint. If I buy a $\frac{1}{8}$ pint sample, it will cover a square two feet long by three feet high. Is the coverage I get more, less or the same as that stated on the label?
6. The world's largest trench digger, "Bagger 288", moves at $\frac{3}{8}$ mph. How long will it take to dig a trench $\frac{2}{3}$ mile long?
7. A $\frac{2}{7}$ Newton force applied to a body of unknown mass produces an acceleration of $\frac{3}{10}$ m/s^2. Calculate the mass of the body.