Equivalence Relation

Subjects to be Learned

- equivalence relation
- equivalence class
- partition

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On the face of most clocks, hours are represented by integers between 1 and 12. However, since a day has 24 hours after 12 hours, a clock goes back to hour 1, and starts all over again from there. Thus each pair of hours such as 1 and 13, 2 and 14, etc. share one number 1, 2, ...etc., respectively. The same applies when we are interested in more than 24 hours. 25th hour is 1, so are 37th, 49th etc. What we are doing here essentially is that we consider the numbers in each group such as 1, 13, 25, ..., equivalent in the sense that they all are represented by one number (they are congruent modulo 12). Being representable by one number such as we see on clocks is a binary relation on the set of natural numbers and it is an example of equivalence relation we are going to study here.

The concept of equivalence relation is characterized by three properties as follows:

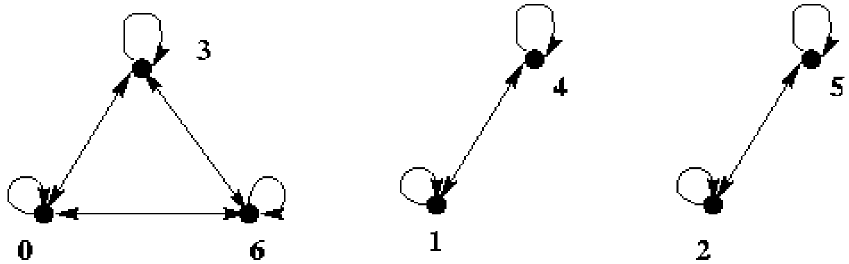
Definition(equivalence relation): A binary relation R on a set A is an **equivalence relation** if and only if
(1) R is reflexive
(2) R is symmetric, and
(3) R is transitive.

Example 1: The equality relation (**=**) on a set of numbers such as {1, 2, 3} is an equivalence relation.
Example 2: The congruent **modulo** m relation on the set of integers i.e. {<a, b>| a $\equiv$ b (**mod** m)}, where m is a positive integer greater than 1, is an equivalence relation.

Note that the equivalence relation on hours on a clock is the congruent **mod** 12, and that when m **=** 2, i.e. the congruent **mod** 2, all even numbers are equivalent and all odd numbers are equivalent. Thus the set of integers are divided into two subsets: evens and odds.

Example 3: Taking this discrete structures course together this semester is another equivalence relation.

Equivalence relations can also be represented by a digraph since they are a binary relation on a set. For example the digraph of the equivalence relation congruent **mod** 3 on {0, 1, 2, 3, 4, 5 , 6} is as shown below. It consists of three connected components.



The set of even numbers and that of odd numbers in the equivalence relation of congruent **mod** 2, and the set of integers equivalent to a number between 1 and 12 in the equivalence relation on hours in the clock example are called an equivalence class. Formally it is defined as follows:

Definition(equivalence class): For an equivalence relation R on a set A, the set of the elements of A that are related to an element, say a, of A is called the **equivalence class** of element a and it is denoted by [a].

Example 4: For the equivalence relation of hours on a clock, equivalence classes are

$[1] = \{1, 13, 25, ...\} = \{1 + 12n : n \in N\}$,

$[2] = \{2, 14, 26, ...\} = \{2 + 12n : n \in N\}$,

........,

where N is the set of natural numbers. There are altogether twelve of them.

For an equivalence relation R on a set A, every element of A is in an equivalence class. For if an element, say b ,does not belong to the equivalence class of any other element in A, then the set consisting of the element b itself is an equivalence class. Thus the set A is in a sense covered by the equivalence classes. Another property of equivalence class is that equivalence classes of two elements of a set A are either disjoint or identical, that is either $[a] = [b]$ or $[a] \cap [b] = \emptyset$ for arbitrary elements a and b of A. Thus the set A is **partitioned** into equivalence classes by an equivalence relation on A. This is formally stated as a theorem below after the definition of partition.

Definition (partition): Let A be a set and let $A_1, A_2, ..., A_n$ be subsets of A. Then $\{A_1, A_2, ..., A_n\}$ is a **partition**of A, if and only if

(1) $\bigcup_{i=1}^{n} A_i = A$, and

(2) $A_i \cap A_j = \emptyset$, if $A_i \neq A_j$, $1 \leq i, j \leq n$.

Example 5: Let $A = \{1, 2, 3, 4, 5\}$, $A_1 = \{1, 5\}$, $A_2 = \{3\}$, and $A_3 = \{2, 4\}$. Then $\{A_1, A_2, A_3\}$ is a partition of A. However, $B_1 = \{1, 2, 5\}$, $B_2 = \{2, 3\}$, and $B_3 = \{4\}$ do not form a partition for A because $B_1 \cap B_2 \neq \emptyset$, though $B_1 \neq B_2$.

Theorem 1: The set of equivalence classes of an equivalence relation on a set A is a partition of A.

Conversely, a partition of a set A determines an equivalence relation on A.

Theorem 2: Let $\{A_1, ..., A_n\}$ be a partition of a set A. Define a binary relation R on A as follows: $<a, b> \in R$ if and only if $a \in A_i$ and $b \in A_i$ for some i, $1 \leq i \leq n$. Then R is an equivalence relation.

Theorem 3: Let R_1 and R_2 be equivalence relations. Then $R_1 \cap R_2$ is an equivalence relation, but $R_1 \cup R_2$ is not necessarily an equivalence relation.

Test Your Understanding of Equivalence Relation

Indicate which of the following statements are correct and which are not.
Click Yes or No , then Submit. There are two sets of questions.