(d) $6-(x-5)+7$
(e) $4(m+7)-6(4-m)$
(f) $-5(y-11)+2y$
(g) $-(x-3y)+\frac{1}{2}(z+4)$
(h) $\frac{a}{b}\left(\frac{2}{a}+\frac{3}{b}+\frac{b}{5}\right)$

3. Use the Distributive Property to simplify the following fractions.
   (a) $\frac{8x+12}{4}$
   (b) $\frac{9x+12}{3}$
   (c) $\frac{11x+12}{2}$
   (d) $\frac{3y+2}{6}$
   (e) $-\frac{6z-2}{3}$
   (f) $\frac{7-6p}{3}$
   (g) $\frac{3d-4}{6d}$
   (h) $\frac{12g+8h}{4gh}$
4. A bookcase has five shelves, and each shelf contains seven poetry books and eleven novels. How many of each type of book does the bookcase contain?
5. Amar is making giant holiday cookies for his friends at school. He makes each cookie with 6 oz of cookie dough and decorates them with macadamia nuts. If Amar has 5 lbs of cookie dough ($1\ lb = 16\ oz$) and 60 macadamia nuts, calculate the following.
   (a) How many (full) cookies he can make?
   (b) How many macadamia nuts he can put on each cookie, if each is to be identical?
   (c) If 4 cups of flour and 1 cup of sugar went into each pound of cookie dough, how much of each did Amar use to make the 5 pounds of dough?

2.5 Square Roots and Real Numbers

Learning Objectives

- Find square roots.
- Approximate square roots.
- Identify irrational numbers.
- Classify real numbers.
- Graph and order real numbers.

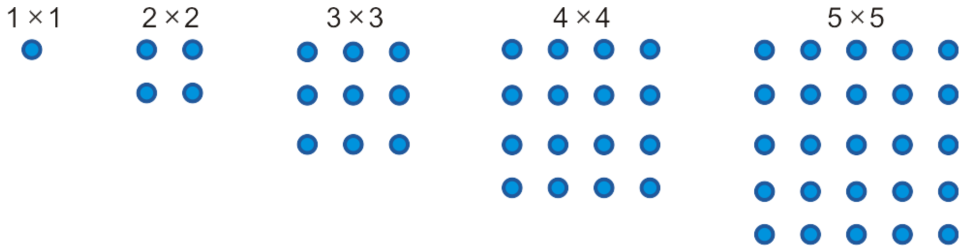
Find Square Roots

The **square root** of a number is a number which, when multiplied by itself, gives the original number. In other words, if $a=b^2$, we say that b is the square root of a.

Note: Negative numbers and positive numbers both yield positive numbers when squared, so each positive number has both a positive and a negative square root. (For example, 3 and -3 can both be squared to yield 9.) The positive square root of a number is called the **principal square root**.

The square root of a number x is written as $\sqrt{x}$ or sometimes as $\sqrt[2]{x}$. The symbol $\sqrt{\ }$ is sometimes called a **radical sign**.

Numbers with whole-number square roots are called **perfect squares**. The first five perfect squares (1, 4, 9, 16, and 25) are shown below.



You can determine whether a number is a perfect square by looking at its prime factors. If every number in the factor tree appears an even number of times, the number is a perfect square. To find the square root of that number, simply take one of each pair of matching factors and multiply them together.

Example 1

Find the principal square root of each of these perfect squares.

a) 121

b) 225

c) 324

d) 576

Solution

a) $121 = 11 \times 11$, so $\sqrt{121} = 11$.

b) $225 = (5 \times 5) \times (3 \times 3)$, so $\sqrt{225} = 5 \times 3 = 15$.

c) $324 = (2 \times 2) \times (3 \times 3) \times (3 \times 3)$, so $\sqrt{324} = 2 \times 3 \times 3 = 18$.

d) $576 = (2 \times 2) \times (2 \times 2) \times (2 \times 2) \times (3 \times 3)$, so $\sqrt{576} = 2 \times 2 \times 2 \times 3 = 24$.

For more practice matching numbers with their square roots, try the Flash games at http://www.quia.com/jg/65631.html.

When the prime factors don't pair up neatly, we "factor out" the ones that do pair up and leave the rest under a radical sign. We write the answer as $a\sqrt{b}$, where a is the product of half the paired factors we pulled out and b is the product of the leftover factors.

Example 2

Find the principal square root of the following numbers.

a) 8

b) 48

c) 75

d) 216

Solution

a) $8 = 2 \times 2 \times 2$. This gives us one pair of 2's and one leftover 2, so $\sqrt{8} = 2\sqrt{2}$.

b) $48 = (2 \times 2) \times (2 \times 2) \times 3$, so $\sqrt{48} = 2 \times 2 \times \sqrt{3}$, or $4\sqrt{3}$.

c) $75 = (5 \times 5) \times 3$, so $\sqrt{75} = 5\sqrt{3}$.

d) $216 = (2 \times 2) \times 2 \times (3 \times 3) \times 3$, so $\sqrt{216} = 2 \times 3 \times \sqrt{2 \times 3}$, or $6\sqrt{6}$.

Note that in the last example we collected the paired factors first, **then** we collected the unpaired ones under a single radical symbol. Here are the four rules that govern how we treat square roots.

- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
- $A\sqrt{a} \times B\sqrt{b} = AB\sqrt{ab}$
- $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
- $\frac{A\sqrt{a}}{B\sqrt{b}} = \frac{A}{B}\sqrt{\frac{a}{b}}$

Example 3

Simplify the following square root problems

a) $\sqrt{8} \times \sqrt{2}$

b) $3\sqrt{4} \times 4\sqrt{3}$

c) $\sqrt{12} \div \sqrt{3}$

d) $12\sqrt{10} \div 6\sqrt{5}$

Solution

a) $\sqrt{8} \times \sqrt{2} = \sqrt{16} = 4$

b) $3\sqrt{4} \times 4\sqrt{3} = 12\sqrt{12} = 12\sqrt{(2 \times 2) \times 3} = 12 \times 2\sqrt{3} = 24\sqrt{3}$

c) $\sqrt{12} \div \sqrt{3} = \sqrt{\frac{12}{3}} = \sqrt{4} = 2$

d) $12\sqrt{10} \div 6\sqrt{5} = \frac{12}{6}\sqrt{\frac{10}{5}} = 2\sqrt{2}$

Approximate Square Roots

Terms like $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{7}$ (square roots of prime numbers) cannot be written as **rational numbers**. That is to say, they cannot be expressed as the ratio of two integers. We call them **irrational numbers**. In decimal form, they have an unending, seemingly random, string of numbers after the decimal point.

To find approximate values for square roots, we use the $\sqrt{}$ or $\sqrt{x}$ button on a calculator. When the number we plug in is a perfect square, or the square of a rational number, we will get an exact answer. When the number is a non-perfect square, the answer will be irrational and will look like a random string of digits. Since the calculator can only show some of the infinitely many digits that are actually in the answer, it is really showing us an **approximate answer**—not exactly the right answer, but as close as it can get.

Example 4

Use a calculator to find the following square roots. Round your answer to three decimal places.

a) $\sqrt{99}$

b) $\sqrt{5}$

c) $\sqrt{0.5}$

d) $\sqrt{1.75}$

Solution

a) ≈ 9.950

b) ≈ 2.236

c) ≈ 0.707

d) ≈ 1.323

You can also work out square roots by hand using a method similar to long division. (See the web page at http://www.homeschoolmath.net/teaching/square-root-algorithm.php for an explanation of this method.)

Identify Irrational Numbers

Not all square roots are irrational, but any square root that can't be reduced to a form with no radical signs in it is irrational. For example, $\sqrt{49}$ is rational because it equals 7, but $\sqrt{50}$ can't be reduced farther than $5\sqrt{2}$. That factor of $\sqrt{2}$ is irrational, making the whole expression irrational.

Example 5

Identify which of the following are rational numbers and which are irrational numbers.

a) 23.7

b) 2.8956

c) π

d) $\sqrt{6}$

e) $3.\overline{27}$

Solution

a) 23.7 can be written as $23\frac{7}{10}$, so it is rational.

b) 2.8956 can be written as $2\frac{8956}{10000}$, so it is rational.

c) $\pi = 3.141592654\ldots$ We know from the definition of π that the decimals do not terminate or repeat, so π is irrational.

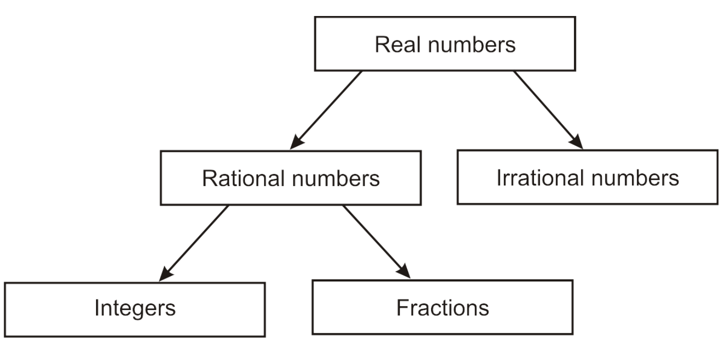
d) $\sqrt{6} = \sqrt{2} \times \sqrt{3}$. We can't reduce it to a form without radicals in it, so it is irrational.

e) $3.\overline{27} = 3.272727272727\ldots$ This decimal goes on forever, but it's not random; it repeats in a predictable pattern. Repeating decimals are always rational; this one can actually be expressed as $\frac{36}{11}$.

You can see from this example that any number whose decimal representation has a finite number of digits is rational, since each decimal place can be expressed as a fraction. For example, 0.439 can be expressed as $\frac{4}{10} + \frac{3}{100} + \frac{9}{1000}$, or just $\frac{439}{1000}$. Also, any decimal that repeats is rational, and can be expressed as a fraction. For example, $0.25\overline{38}$ can be expressed as $\frac{25}{100} + \frac{38}{9900}$, which is equivalent to $\frac{2513}{9900}$.

Classify Real Numbers

We can now see how real numbers fall into one of several categories.



If a real number can be expressed as a rational number, it falls into one of two categories. If the denominator of its **simplest form** is one, then it is an **integer**. If not, it is a fraction (this term also includes decimals, since they can be written as fractions.)

If the number cannot be expressed as the ratio of two integers (i.e. as a fraction), it is **irrational**.

Example 6

Classify the following real numbers.

a) 0

b) -1

c) $\frac{\pi}{3}$

d) $\frac{\sqrt{2}}{3}$

e) $\frac{\sqrt{36}}{9}$

Solution

a) Integer

b) Integer

c) Irrational (Although it's written as a fraction, π is irrational, so any fraction with π in it is also irrational.)

d) Irrational

e) Rational (It simplifies to $\frac{6}{9}$, or $\frac{2}{3}$.)

Lesson Summary

- The **square root** of a number is a number which gives the original number when multiplied by itself. In algebraic terms, for two numbers a and b, if $a = b^2$, then $b = \sqrt{a}$.
- A square root can have two possible values: a positive value called the **principal square root**, and a negative value (the opposite of the positive value).
- A **perfect square** is a number whose square root is an integer.
- Some mathematical properties of square roots are:
  - $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
  - $A\sqrt{a} \times B\sqrt{b} = AB\sqrt{ab}$
  - $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
  - $\frac{A\sqrt{a}}{B\sqrt{b}} = \frac{A}{B}\sqrt{\frac{a}{b}}$
- Square roots of numbers that are not perfect squares (or ratios of perfect squares) are **irrational numbers**. They cannot be written as rational numbers (the ratio of two integers). In decimal form, they have an unending, seemingly random, string of numbers after the decimal point.
- Computing a square root on a calculator will produce an **approximate solution** since the calculator only shows a finite number of digits after the decimal point.

Review Questions

1. Find the following square roots **exactly without using a calculator**, giving your answer in the simplest form.
   (a) $\sqrt{25}$
   (b) $\sqrt{24}$

(c) $\sqrt{20}$
(d) $\sqrt{200}$
(e) $\sqrt{2000}$
(f) $\sqrt{\frac{1}{4}}$ (Hint: The division rules you learned can be applied backwards!)
(g) $\sqrt{\frac{9}{4}}$
(h) $\sqrt{0.16}$
(i) $\sqrt{0.1}$
(j) $\sqrt{0.01}$

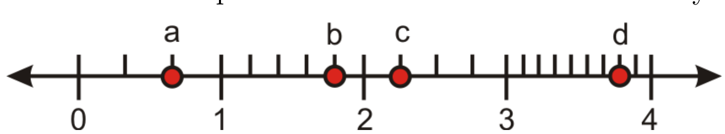
2. Use a calculator to find the following square roots. Round to two decimal places.

(a) $\sqrt{13}$
(b) $\sqrt{99}$
(c) $\sqrt{123}$
(d) $\sqrt{2}$
(e) $\sqrt{2000}$
(f) $\sqrt{.25}$
(g) $\sqrt{1.35}$
(h) $\sqrt{0.37}$
(i) $\sqrt{0.7}$
(j) $\sqrt{0.01}$

3. Classify the following numbers as an integer, a rational number or an irrational number.

(a) $\sqrt{0.25}$
(b) $\sqrt{1.35}$
(c) $\sqrt{20}$
(d) $\sqrt{25}$
(e) $\sqrt{100}$

4. Place the following numbers in numerical order, from lowest to highest.
$\frac{\sqrt{6}}{2}$ $\quad \frac{61}{50}$ $\quad \sqrt{1.5}$ $\quad \frac{16}{13}$
5. Use the marked points on the number line and identify each proper fraction.



2.6 Problem-Solving Strategies: Guess and Check, Work Backward

Learning Objectives

- Read and understand given problem situations.
- Develop and use the strategy "Guess and Check."
- Develop and use the strategy "Work Backward."
- Plan and compare alternative approaches to solving problems.
- Solve real-world problems using selected strategies as part of a plan.