CHAPTER 1

Domain and Range of a Function

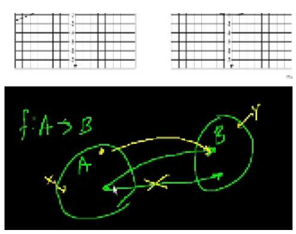
Here you'll learn how to find the domain and range of a function and you'll make a table of values for a given function.

What if you were given a rule that relates two variables, like $f(x) = 5x^2 + 1$? How could you find the domain and range of the function defined by that rule? After completing this Concept, you'll be able to identify the domain and range of functions like this one.

Watch This

CK-12 Foundation: 0110S Introductionto Functions

For another look at the domain of a function, see the following video, where the narrator solves a sample problem from the California Standards Test about finding the domain of an unusual function.



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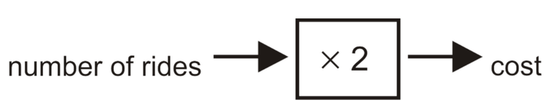
KhanAcademy: CAAlgebraI: Functions

Guidance

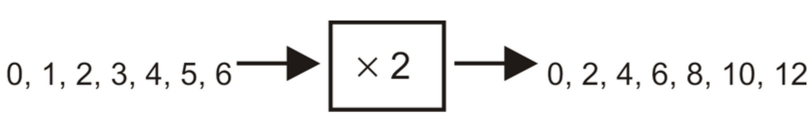
A **function** is a rule for relating two or more variables. For example, the price you pay for phone service may depend on the number of minutes you talk on the phone. We would say that the cost of phone service is a *function* of the number of minutes you talk. Consider the following situation.

Josh goes to an amusement park where he pays $2 per ride.

There is a relationship between the number of rides Josh goes on and the total amount he spends that day: To figure out the amount he spends, we multiply the number of rides by two. This rule is an example of a **function**. Functions usually—*but not always*—are rules based on mathematical operations. You can think of a function as a box or a machine that contains a mathematical operation.



Whatever number we feed into the function box is changed by the given operation, and a new number comes out the other side of the box. When we input different values for the number of rides Josh goes on, we get different values for the amount of money he spends.

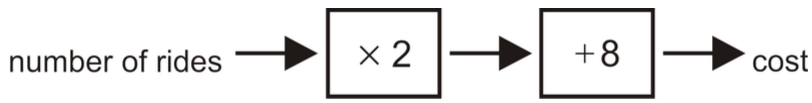


The input is called the **independent variable** because its value can be any number. The output is called the **dependent variable** because its value depends on the input value.

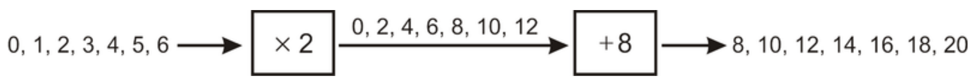
Functions usually contain more than one mathematical operation. Here is a situation that is slightly more complicated than the example above.

Jason goes to an amusement park where he pays $8 admission and $2 per ride.

The following function represents the total amount Jason pays. The rule for this function is "multiply the number of rides by 2 and add 8."



When we input different values for the number of rides, we arrive at different outputs (costs).



These flow diagrams are useful in visualizing what a function is. However, they are cumbersome to use in practice. In algebra, we use the following short-hand notation instead:

$$\begin{array}{c} input \\ \downarrow \\ \underbrace{f(x)}_{\substack{function \\ box}} = y \leftarrow output \end{array}$$

First, we define the variables:

$x =$ the number of rides Jason goes on

$y =$ the total amount of money Jason spends at the amusement park.

So, x represents the input and y represents the output. The notation $f()$ represents the function or the mathematical operations we use on the input to get the output. In the last example, the cost is 2 times the number of rides plus 8. This can be written as a function:

$$f(x) = 2x + 8$$

In algebra, the notations y and $f(x)$ are typically used interchangeably. Technically, though, $f(x)$ represents the function itself and y represents the output of the function.

Identify the Domain and Range of a Function

In the last example, we saw that we can input the number of rides into the function to give us the total cost for going to the amusement park. The set of all values that we can use for the input is called the **domain** of the function, and the set of all values that the output could turn out to be is called the **range** of the function. In many situations the **domain** and **range** of a function are both simply the set of all real numbers, but this isn't always the case. Let's look at our amusement park example.

Example A

Find the domain and range of the function that describes the situation:

Jason goes to an amusement park where he pays $8 admission and $2 per ride.

Solution

Here is the function that describes this situation:

$$f(x) = 2x + 8 = y$$

In this function, x is the number of rides and y is the total cost. To find the domain of the function, we need to determine which numbers make sense to use as the input (x).

- The values have to be zero or positive, because Jason can't go on a negative number of rides.
- The values have to be integers because, for example, Jason could not go on 2.25 rides.
- Realistically, there must be a maximum number of rides that Jason can go on because the park closes, he runs out of money, etc. However, since we aren't given any information about what that maximum might be, we must consider that all non-negative integers are possible values regardless of how big they are.

Answer For this function, the domain is the set of all non-negative integers.

To find the range of the function we must determine what the values of y will be when we apply the function to the input values. The domain is the set of all non-negative integers: 0, 1, 2, 3, 4, 5, 6, ... Next we plug these values into the function for x. If we plug in 0, we get 8; if we plug in 1, we get 10; if we plug in 2, we get 12, and so on, counting by 2s each time. Possible values of y are therefore 8, 10, 12, 14, 16, 18, 20... or in other words all even integers greater than or equal to 8.

Answer The range of this function is the set of all even integers greater than or equal to 8.

Example B

Find the domain and range of the following functions.

a) A ball is dropped from a height and it bounces up to 75% of its original height.

b) $y = x^2$

Solution

a) Let's define the variables:

$x =$ original height

$y =$ bounce height

A function that describes the situation is $y = f(x) = 0.75x$. x can represent any real value greater than zero, since you can drop a ball from any height greater than zero. A little thought tells us that y can also represent any real value greater than zero.

Answer

The domain is the set of all real numbers greater than zero. The range is also the set of all real numbers greater than zero.

b) Since there is no word problem attached to this equation, we can assume that we can use any real number as a value of x. When we square a real number, we always get a non-negative answer, so y can be any non-negative real number.

Answer

The domain of this function is all real numbers. The range of this function is all non-negative real numbers.

In the functions we've looked at so far, x is called the **independent variable** because it can be any of the values from the domain, and y is called the **dependent variable** because its value depends on x. However, any letters or symbols can be used to represent the dependent and independent variables. Here are three different examples:

$$\begin{aligned} y &= f(x) = 3x \\ R &= f(w) = 3w \\ v &= f(t) = 3t \end{aligned}$$

These expressions all represent the same function: a function where the dependent variable is three times the independent variable. Only the symbols are different. In practice, we usually pick symbols for the dependent and independent variables based on what they represent in the real world—like t for time, d for distance, v for velocity, and so on. But when the variables don't represent anything in the real world—or even sometimes when they do—we traditionally use y for the dependent variable and x for the independent variable.

Make a Table For a Function

A table is a very useful way of arranging the data represented by a function. We can match the input and output values and arrange them as a table. For example, the values from Example 1 above can be arranged in a table as follows:

x	0	1	2	3	4	5	6
y	8	10	12	14	16	18	20

A table lets us organize our data in a compact manner. It also provides an easy reference for looking up data, and it gives us a set of coordinate points that we can plot to create a graph of the function.

Example C

Make a table of values for the function $f(x) = \frac{1}{x}$. Use the following numbers for input values: -1, -0.5, -0.2, -0.1, -0.01, 0.01, 0.1, 0.2, 0.5, 1.

Solution

Make a table of values by filling the first row with the input values and the next row with the output values calculated using the given function.

x	-1	-0.5	-0.2	-0.1	-0.01	0.01	0.1	0.2	0.5	1
$f(x) = \frac{1}{x}$	$\frac{1}{-1}$	$\frac{1}{-0.5}$	$\frac{1}{-0.2}$	$\frac{1}{-0.1}$	$\frac{1}{-0.01}$	$\frac{1}{0.01}$	$\frac{1}{0.1}$	$\frac{1}{0.2}$	$\frac{1}{0.5}$	$\frac{1}{1}$
y	-1	-2	-5	-10	-100	100	10	5	2	1

When you're given a function, you won't usually be told what input values to use; you'll need to decide for yourself what values to pick based on what kind of function you're dealing with. We will discuss how to pick input values throughout these lessons.

Watch this video for help with the Examples above.

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Vocabulary

A **function** is a rule for relating two or more variables, one of which is the input variable and the other is the output variable. The input is called the **independent variable** because its value can be any number. The output is called the **dependent variable** because its value depends on the input value. The set of all values that we can use for the input is called the **domain** of the function, and the set of all values that the output could turn out to be is called the **range** of the function.

Guided Practice

Identify the domain and then make a table of values for the function $f(x) = \frac{1}{\sqrt{x}}$. Use the following numbers for input values: 0.01, 0.16, 0.25, 1, 4.

Solution

Since you cannot compute the square root of negative numbers, these cannot be in the domain. Since we cannot have 0 in the denominator, 0 is also not in the domain. This means that the domain is all real numbers greater than zero.

Make a table of values by filling the first row with the input values and the next row with the output values calculated using the given function.

x	0.01	0.16	0.25	1	4
$f(x) = \frac{1}{x}$	$\frac{1}{\sqrt{0.01}}$	$\frac{1}{\sqrt{0.16}}$	$\frac{1}{\sqrt{0.25}}$	$\frac{1}{\sqrt{1}}$	$\frac{1}{\sqrt{4}}$
y	10	2.5	2	1	0.5

Practice

For 1-6, identify the domain and range of the following functions.

1. Dustin charges $10 per hour for mowing lawns.
2. Maria charges $25 per hour for tutoring math, with a minimum charge of $15.
3. $f(x) = 15x - 12$
4. $f(x) = 2x^2 + 5$
5. $f(x) = \frac{1}{x}$
6. $f(x) = \sqrt[3]{x}$
7. What is the range of the function $y = x^2 - 5$ when the domain is -2, -1, 0, 1, 2?
8. What is the range of the function $y = 2x - \frac{3}{4}$ when the domain is -2.5, -1.5, 5?
9. What is the domain of the function $y = 3x$ when the range is 9, 12, 15?
10. What is the range of the function $y = 3x$ when the domain is 9, 12, 15?
11. Angie makes $6.50 per hour working as a cashier at the grocery store. Make a table that shows how much she earns if she works 5, 10, 15, 20, 25, or 30 hours.
12. The area of a triangle is given by the formula $A = \frac{1}{2}bh$. If the base of the triangle measures 8 centimeters, make a table that shows the area of the triangle for heights 1, 2, 3, 4, 5, and 6 centimeters.
13. Make a table of values for the function $f(x) = \sqrt{2}x + 3$ for input values -1, 0, 1, 2, 3, 4, 5.

CHAPTER 2 Function Notation

Here you'll learn how to take an equation or inequality and rewrite it as a function. You'll also find out the difference between an independent variable and a dependent variable.

Suppose that you want to set up a function that allows you to input a dog's age in human years and which outputs the dog's age in dog years. How would you go about setting up such a function, and what notation would you use? Would the notation be the same as it was for the equations that you've looked at in previous Concepts? In this Concept, you'll learn what is needed to write a function such as this.

Guidance

Instead of purchasing a one-day ticket to the theme park, Joseph decided to pay by ride. Each ride costs \$2.00. To describe the amount of money Joseph will spend, several mathematical concepts can be used.

First, an expression can be written to describe the relationship between the cost per ride and the number of rides, r. An equation can also be written if the total amount he wants to spend is known. An inequality can be used if Joseph wanted to spend less than a certain amount.

Example A

Using Joseph's situation, write the following:

a. An expression representing his total amount spent

b. An equation representing his total amount spent

c. An equation that shows Joseph wants to spend exactly \$22.00 on rides

d. An inequality that describes the fact that Joseph will not spend more than \$26.00 on rides

Solution: The variable in this situation is the number of rides Joseph will pay for. Call this r.

a. $2(r)$

b. $2(r) = m$

c. $2(r) = 22$

d. $2(r) \leq 26$

In addition to an expression, equation, or inequality, Joseph's situation can be expressed in the form of a function or a table.

Definition: A **function** is a relationship between two variables such that the input value has ONLY one output value.

Writing Equations as Functions

A function is a set of ordered pairs in which the first coordinate, usually x, matches with exactly one second coordinate, y. Equations that follow this definition can be written in function notation. The y coordinate represents the **dependent variable**, meaning the values of this variable depend upon what is substituted for the other variable.

Consider Joseph's equation $m = 2r$. Using function notation, the value of the equation (the money spent m) is replaced with $f(r)$. f represents the function name and (r) represents the variable. In this case the parentheses do not mean multiplication; rather, they separate the function name from the **independent variable.**

$$\begin{array}{c} input \\ \downarrow \\ \underbrace{f(x)}_{\substack{function \\ box}} = y \leftarrow output \end{array}$$

Example B

Rewrite the following equations in function notation.

a. $y = 7x - 3$

b. $d = 65t$

c. $F = 1.8C + 32$

Solution:

a. According to the definition of a function, $y = f(x)$, so $f(x) = 7x - 3$.

b. This time the dependent variable is d. Function notation replaces the dependent variable, so $d = f(t) = 65t$.

c. $F = f(C) = 1.8C + 32$

Why Use Function Notation?

Why is it necessary to use function notation? The necessity stems from using multiple equations. Function notation allows one to easily decipher between the equations. Suppose Joseph, Lacy, Kevin, and Alfred all went to the theme park together and chose to pay \$2.00 for each ride. Each person would have the same equation $m = 2r$. Without asking each friend, we could not tell which equation belonged to whom. By substituting function notation for the dependent variable, it is easy to tell which function belongs to whom. By using function notation, it will be much easier to graph multiple lines.

Example C

Write functions to represent the total each friend spent at the park.

Solution:

$J(r) = 2r$ represents Joseph's total,

$L(r) = 2r$ represents Lacy's total,

$K(r) = 2r$ represents Kevin's total, and

$A(r) = 2r$ represents Alfred's total.

Vocabulary

Function: A ***function*** is a relationship between two variables such that the input value has ONLY one output value.

Dependent variable: A ***dependent variable*** is one whose values depend upon what is substituted for the other variable.

Independent variable: The ***independent variable*** is the variable which is not dependent on another variable. The dependent variable is dependent on the independent variable.

Guided Practice

Recall the example from a previous Concept where a student organization sells shirts to raise money. The cost of printing the shirts was expressed as $100+7x$ and for the revenue, we had the expression $15x$, where x is the number of shirts.

a. Write two functions, one for the cost and one for revenue.

b. Express that the cost must be less than or equal to $800.

c. Express that the revenue must be equal to $1500.

d. How many shirts must the students sell in order to make $1500?

Solution:

a. The cost function we will write as $C(x) = 100+7x$ and the revenue function we will write as $R(x) = 15x$.

b. Since $C(x)$ represents the costs, we substitute in $800 for $C(x)$ and replace the equation with the appropriate inequality symbol

$$100+7x \leq 800$$

This reads that $100+7x$ is less than or equal to $800, so we have written the inequality correctly.

c. We substitute in $1500 for $R(x)$, getting

$1500 = 15x$.

d. We want to find the value of x that will make this equation true. It looks like 100 is the answer. Checking this we see that 100 does satisfy the equation. The students must sell 100 shirts in order to have a revenue of $1500.

$1500 = 15(100)$

$1500 = 1500$

Practice

1. Rewrite using function notation: $y = \frac{5}{6}x - 2$.
2. Rewrite using function notation: $m = n^2 + 2n - 3$.
3. What is one benefit of using function notation?
4. Write a function that expresses the money earned after working some number of hours for $10 an hour.

5. Write a function that represents the number of cuts you need to cut a ribbon in x number of pieces.
6. Jackie and Mayra each will collect a $2 pledge for every basket they make during a game. Write two functions, one for each girl, expressing how much money she will collect.

Mixed Review

7. Compare the following numbers 23 __ 21.999.
8. Write an equation to represent the following: the quotient of 96 and 4 is g.
9. Write an inequality to represent the following: 11 minus b is at least 77.
10. Find the value of the variable $k : 13(k) = 169$.