# Chapter 6

# Logarithms

## 6.1 Logarithm Concepts[1]

Suppose you are a biologist investigating a population that doubles every year. So if you start with 1 specimen, the population can be expressed as an exponential function: $p(t) = 2^t$ where $t$ is the number of years you have been watching, and $p$ is the population.

Question: How long will it take for the population to exceed 1,000 specimens?

We can rephrase this question as: "2 to **what power** is 1,000?" This kind of question, where you know the base and are looking for the exponent, is called a **logarithm**.

$\log_2 1000$ (read, "the logarithm, base two, of a thousand") means "2, raised to what power, is 1000?"

In other words, the logarithm always asks "**What exponent should we use**?" This unit will be an exploration of logarithms.

### 6.1.1 A few quick examples to start things off

| Problem | Means | The answer is | because |
|---|---|---|---|
| $\log_2 8$ | 2 to what power is 8? | 3 | $2^3$ is 8 |
| $\log_2 16$ | 2 to what power is 16? | 4 | $2^4$ is 16 |
| $\log_2 10$ | 2 to what power is 10? | somewhere between 3 and 4 | $2^3 = 8$ and $2^4 = 16$ |
| $\log_8 2$ | 8 to what power is 2? | $\frac{1}{3}$ | $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$ |
| $\log_{10} 10,000$ | 10 to what power is 10,000? | 4 | $10^4 = 10,000$ |
| $\log_{10}\left(\frac{1}{100}\right)$ | 10 to what power is $\frac{1}{100}$ ? | –2 | $10^{--2} = \frac{1}{10^2} = \frac{1}{100}$ |
| $\log_5 0$ | 5 to what power is 0? | There is no answer | $5^{\text{something}}$ will never be 0 |

**Table 6.1**

As you can see, one of the most important parts of finding logarithms is being very familiar with how exponents work!

[1]This content is available online at <http://cnx.org/content/m18242/1.5/>.

## 6.2 The Logarithm Explained by Analogy to Roots[2]

The logarithm may be the first really new concept you've encountered in Algebra II. So one of the easiest ways to understand it is by comparison with a familiar concept: roots.

Suppose someone asked you: "Exactly what does root mean?" You do understand roots, but they are difficult to define. After a few moments, you might come up with a definition very similar to the "question" definition of logarithms given above. $\sqrt[3]{8}$ means "what number cubed is 8?"

Now the person asks: "How do you find roots?" Well...you just play around with numbers until you find one that works. If someone asks for $\sqrt{25}$, you just have to know that $5^2 = 25$. If someone asks for $\sqrt{30}$, you know that has to be bigger than 5 and smaller than 6; if you need more accuracy, it's time for a calculator.

All that information about roots applies in a very analogous way to logarithms.

| | **Roots** | **Logs** |
|---|---|---|
| The question | $\sqrt[a]{x}$ means "what number, raised to the a power, is x?" As an equation, $?^a = x$ | $\log_a x$ means " $a$, raised to what power, is $x$?" As an equation, $a^? = x$ |
| Example that comes out even | $\sqrt[3]{8} = 2$ | $\log_2 8 = 3$ |
| Example that doesn't | $\sqrt[3]{10}$ is a bit more than 2 | $\log_2 10$ is a bit more than 3 |
| Out of domain example | $\sqrt{-4}$does not exist ( $x^2$ will never give $-4$) | $\log_2(0)$ and $\log_2(-1)$ do not exist ( $2^x$ will never give 0 or a negative answer) |

**Table 6.2**

## 6.3 Rewriting Logarithm Equations as Exponent Equations[3]

Both root equations and logarithm equations can be rewritten as exponent equations.

$\sqrt{9} = 3$ can be rewritten as $3^2 = 9$. These two equations are the same statement about numbers, written in two different ways. $\sqrt{9}$ asks the question "What number squared is 9?" So the equation $\sqrt{9} = 3$asks this question, and then answers it: "3 squared is 9."

We can rewrite logarithm equations in a similar way. Consider this equation:

$$\log_3\left(\frac{1}{3}\right) = -1 \tag{6.1}$$

If you are asked to rewrite that logarithm equation as an exponent equation, think about it this way. The left side asks: "3 to what power is $\left(\frac{1}{3}\right)$?" And the right side answers: "3 to the $-1$**power** is $\left(\frac{1}{3}\right)$." $3^{-1} = \left(\frac{1}{3}\right)$.

[2]This content is available online at <http://cnx.org/content/m18236/1.2/>.
[3]This content is available online at <http://cnx.org/content/m18241/1.3/>.

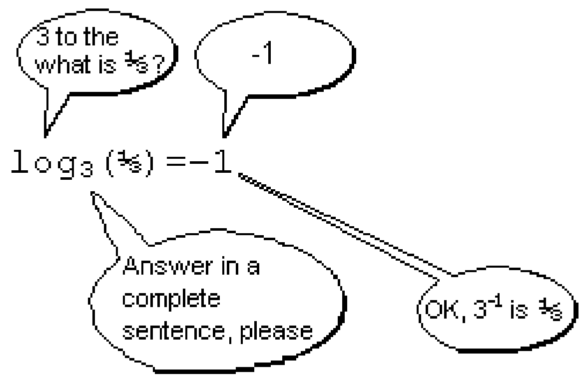


**Figure 6.1**

These two equations, $\log_3\left(\frac{1}{3}\right) = -1$and $3^{-1} = \left(\frac{1}{3}\right)$, are two different ways of expressing the same numerical relationship.

## 6.4 The Logarithm Defined as an Inverse Function[4]

$\sqrt{x}$ can be defined as the **inverse function** of $x^2$. Recall the definition of an inverse function— $f^{-1}(x)$ is defined as the inverse of $f^1(x)$ if it reverses the inputs and outputs. So we can demonstrate this inverse relationship as follows:

| $\sqrt{x}$ **is the inverse function of** $x^2$ |
| --- |
| $3 \rightarrow x^2 \rightarrow 9$ |
| $9 \rightarrow \sqrt{x} \rightarrow 3$ |

**Table 6.3**

Similarly, $\log_2 x$ is the **inverse function** of the exponential function $2^x$.

| $\log_2 x$ **is the inverse function of** $2^x$ |
| --- |
| $3 \rightarrow 2^x \rightarrow 8$ |
| $8 \rightarrow \log_2 x \rightarrow 2$ |

**Table 6.4**

(You may recall that during the discussion of inverse functions, $2^x$ was the only function you were given that you **could not find** the inverse of. Now you know!)

In fact, as we noted in the first chapter, $\sqrt{x}$is **not** a perfect inverse of $x^2$, since it does not work for negative numbers. $(-3)^2 = 9$, but $\sqrt{9}$ is not $-3$. Logarithms have no such limitation: $\log_2 x$ is a perfect inverse for $2^x$.

[4]This content is available online at <http://cnx.org/content/m18240/1.3/>.

The inverse of addition is subtraction. The inverse of multiplication is division. Why do exponents have two completely different kinds of inverses, roots and logarithms? **Because exponents do not commute**. $3^2$ and $2^3$ are not the same number. So the question "what number squared equals 10?" and the question "2 to what power equals 10?" are different questions, which we express as $\sqrt{10}$ and $\log_2 10$, respectively, and they have different answers. $x^2$ and $2^x$ are not the same function, and they therefore have different inverse functions $\sqrt{x}$ and $\log_2 10$.

## 6.5 Properties of Logarithms[5]

Just as there are three fundamental laws of exponents, there are three fundamental laws of logarithms.

$$\log_x (\mathrm{ab}) = \log_x a + \log_x b \tag{6.2}$$

$$\log_x \frac{a}{b} = \log_x a - \log_x b \tag{6.3}$$

$$\log_x \left(a^b\right) = b\log_x a \tag{6.4}$$

As always, these **algebraic generalizations** hold for any $a$, $b$, and $x$.

**Example 6.1: Properties of Logarithms**

1. **Suppose you are given these two facts**:
   $\log_4 5 = 1.16$
   $\log_4 10 = 1.66$
2. **Then we can use the laws of logarithms to conclude that**:
   $\log_4 (50) = \log_4 5 + \log_4 10 = 2.82$
   $\log_4 (2) = \log_4 10 - \log_4 5 = 0.5$
   $\log_4 (100{,}000) = 5\log_4 10 = 8.3$

NOTE: **All three of these results can be found quickly, and without a calculator. Note that the second result could also be figured out directly, since** $4^{\frac{1}{2}} = 2$.

These properties of logarithms were very important historically, because they enabled pre-calculator mathematicians to perform **multiplication** (which is very time-consuming and error prone) by doing **addition** (which is faster and easier). These rules are still useful in simplifying complicated expressions and solving equations.

**Example 6.2: Solving an equation with the properties of logarithms**

| | | |
|---|---|---|
| $\log_2 x - \log_x (x-1) = 5$ | **The problem** | |
| $\log_2 \left(\frac{x}{x-1}\right) = 5$ | **Second property of logarithms** | |
| $\frac{x}{x-1} = 2^5 = 32$ | **Rewrite the log as an exponent. (2-to-what is? $\frac{x}{x-1}$ 2-to-the-5!)** | |
| $x = 32(x-1)$ | **Multiply. We now have an easy equation to solve.** | |
| $x = 32x - 32$ | | |
| $-31x = -32$ | | |
| $x = \frac{32}{31}$ | | |

[5]This content is available online at <http://cnx.org/content/m18239/1.4/>.

**Table 6.5**

### 6.5.1 Proving the Properties of Logarithms

If you understand what an exponent is, you can very quickly see why the three rules of exponents work. But why do logarithms have these three properties?

As you work through the text, you will demonstrate these rules intuitively, by viewing the logarithm as a **counter**. ( $\log_2 8$ asks "**how many** 2s do I need to multiply, in order to get 8?") However, these rules can also be rigorously proven, using the laws of exponents as our starting place.

| Proving the First Law of Logarithms, $\log_x(\text{ab}) = \log_x a + \log_x b$ | |
|---|---|
| $m = \log_x a$ | **I'm just inventing $m$ to represent this log** |
| $x^m = a$ | **Rewriting the above expression as an exponent. ( $\log_x a$ asks " $x$ to what power is $a$ ?" And the equation answers: " $x$ to the $m$ is $a$ .")** |
| $n = \log_x b$ | **Similarly, $n$ will represent the other log.** |
| $x^n = b$ | |
| $\log_x(\text{ab}) = \log_x(x^m x^n)$ | **Replacing $a$ and $b$ based on the previous equations** |
| $= \log_x(x^{m+n})$ | **This is the key step! It uses the first law of exponents. Thus you can see that the properties of logarithms come directly from the laws of exponents.** |
| $= m + n$ | **$= \log_x(x^{m+n})$ asks the question: " $x$ to what power is $x^{m+n}$ ?" Looked at this way, the answer is obviously $(m+n)$ . Hence, you can see how the logarithm and exponential functions cancel each other out, as inverse functions must.** |
| $= \log_x a + \log_x b$ | **Replacing $m$ and $n$ with what they were originally defined as. Hence, we have proven what we set out to prove.** |

**Table 6.6**

To test your understanding, try proving the second law of logarithms: the proof is very similar to the first. For the third law, you need invent only one variable, $m = \log_x a$. In each case, you will rely on a different one of the three rules of exponents, showing how each exponent law corresponds to one of the logarithms laws.

## 6.6 Common Logarithms[6]

When you see a root without a number in it, it is assumed to be a **square** root. That is, $\sqrt{25}$is a shorthand way of writing $\sqrt[2]{25}$. This rule is employed because **square** roots are more common than other types.

[6]This content is available online at <http://cnx.org/content/m18237/1.4/>.

When you see a logarithm without a number in it, it is assumed to be a **base 10** logarithm. That is, $\log(1000)$ is a shorthand way of writing $\log_{10}(1000)$. A base 10 logarithm is also known as a "common" log.

Why are common logs particularly useful? Well, what is $\log_{10}(1000)$? By now you know that this asks the question "10 to what power is 1000?" The answer is 3. Similarly, you can confirm that:

$$\log(10) = 1 \tag{6.5}$$

$$\log(100) = 2 \tag{6.6}$$

$$\log(1,000,000) = 6 \tag{6.7}$$

We can also follow this pattern backward:

$$\log(1) = 0 \tag{6.8}$$

$$\log\left(\frac{1}{10}\right) = -1 \tag{6.9}$$

$$\log\left(\frac{1}{100}\right) = -2 \tag{6.10}$$

and so on. In other words, the common log tells you the **order of magnitude** of a number: how many zeros it has. Of course, $\log_{10}(500)$ is difficult to determine exactly without a calculator, but we can say immediately that it must be somewhere between 2 and 3, since 500 is between 100 and 1000.

## 6.7 Graphing Logarithmic Functions[7]

Suppose you want to graph the function $y = \log_2(x)$. You might start by making a table that looks something like this:

| $x$ | $y = \mathbf{log_2}(x)$ |
|---|---|
| 1 | 0 |
| 2 | 1 |
| 3 | um....I'm not sure |
| 4 | 2 |
| 5 | can I use a calculator? |

**Table 6.7**

This doesn't seem to be the right strategy. Many of those numbers are just too hard to work with.

So, you start looking for numbers that **are** easy to work with. And you remember that it's important to look at numbers that are less than 1, as well as greater. And eventually, you end up with something more like this.

[7] This content is available online at <http://cnx.org/content/m18238/1.4/>.

| $x$ | $y = \mathbf{log}_2(x)$ |
|---|---|
| $\frac{1}{8}$ | $-3$ |
| $\frac{1}{4}$ | $-2$ |
| $\frac{1}{2}$ | $-1$ |
| 1 | 0 |
| 2 | 1 |
| 4 | 2 |
| 8 | 3 |

**Table 6.8**

As long as you keep putting powers of 2 in the $x$ column, the $y$ column is very easy to figure.

In fact, the easiest way to generate this table is to recognize that it is the table of $y = 2^x$ values, only with the $x$ and $y$ coordinates switched! In other words, we have re-discovered what we already knew: that $y = 2^x$ and $y = \log_2(x)$ are inverse functions.

When you graph it, you end up with something like this:

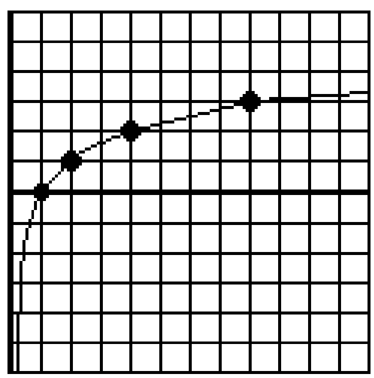

**Figure 6.2:** $y = \log_2(x)$

As always, you can learn a great deal about the log function by reading the graph.

- The domain is $x > 0$. (You can't take the log of 0 or a negative number—do you remember why?).
- The range, on the other hand, is all numbers. Of course, all this inverses the function $2^x$, which has a **domain** of all numbers and a **range** of $y > 0$.
- As $x$ gets closer and closer to 0, the function dives down to smaller and smaller negative numbers. So the $y$-axis serves as an "asymptote" for the graph, meaning a line that the graph **approaches closer and closer to** without ever touching.
- As $x$ moves to the right, the graph grows—but more and more slowly. As $x$ goes from 4 to 8, the graph goes up by 1. As $x$ goes from 8 to 16, the graph goes up by another 1. It doesn't make it up another 1 until $x$ reaches 32...and so on.

This pattern of **slower and slower growth** is one of the most important characteristics of the log. It can be used to "slow down" functions that have too wide a range to be practical to work with.

**Example 6.3: Using the log to model a real world problem**
Lewis Fry Richardson (1881–1953) was a British meteorologist and mathematician. He was also an active Quaker and committed pacifist, and was one of the first men to apply statistics to the study of human conflict. Richardson catalogued 315 wars between 1820 and 1950, and categorized them by **how many deaths** they caused. At one end of the scale is a deadly quarrel, which might result in 1 or 2 deaths. At the other extreme are World War I and World War II, which are responsible for roughly 10 million deaths each.

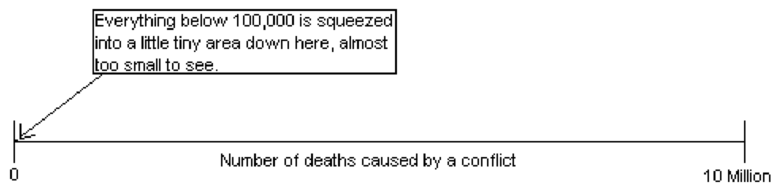


**Figure 6.3**

As you can see from the chart above, working with these numbers is extremely difficult: on a scale from 0 to 10 Million, there is no visible difference between (say) 1 and 100,000. Richardson solved this problem by taking the **common log of the number of deaths**. So a conflict with 1,000 deaths is given a magnitude of $\log(1000) = 3$. On this scale, which is now the standard for conflict measurement, the magnitudes of all wars can be easily represented.

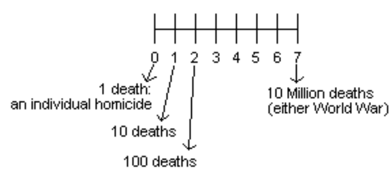


**Figure 6.4**

Richardson's scale makes it practical to chart, discuss, and compare wars and battles from the smallest to the biggest. For instance, he discovered that each time you move up by one on the scale—that is, each time the number of deaths multiplies by 10—the number of conflicts drops in a third. (So there are roughly three times as many "magnitude 5" wars as "magnitude 6," and so on.)

The log is useful here because the logarithm function itself **grows so slowly** that it compresses the entire 1-to-10,000,000 range into a 0-to-7 scale. As you will see in the text, the same trick is used—for the same reason—in fields ranging from earthquakes to sound waves.