Even and odd functions*

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Even and odd functions are related to symmetry of functions. The symmetry of a function is visualized by the planar plot of a function, which may show symmetry with respect to either an axis (y-axis) or origin.

Since functions need not always be symmetric, they may neither be even nor be odd. The parity of a function i.e. whether it is even or odd is determined with certain algebraic algorithm. Further, symmetry of functions may change subsequent to mathematical operations.

1 Even functions

The values of even function at x=x and x=-x are same.

Definition 1: Even function
A function f(x) is said to be "even" if for every "x", there exists "-x" in the domain of the function such that :

$$f(-x) = f(x)$$

An even function is symmetric about y-axis. If we consider the axis as a mirror, then the plot in first quadrant has its mirror image (bilaterally inverted) in second quadrant. Similarly, the plot in fourth quadrant has its mirror image (bilaterally inverted) in third quadrant.

Some examples of even functions are $x^2, |x| \quad \text{and} \quad \cos x$. In each case, we see that :

$$\Rightarrow f(-x) = (-x)^2 = x^2 = f(x)$$

$$\Rightarrow f(-x) = |-x| = |x| = f(x)$$

$$\Rightarrow f(-x) = \cos(-x) = \cos x = f(x)$$

The right side is mirror image of left hand side and the left side is mirror image of right hand side of the curve.

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Even functions

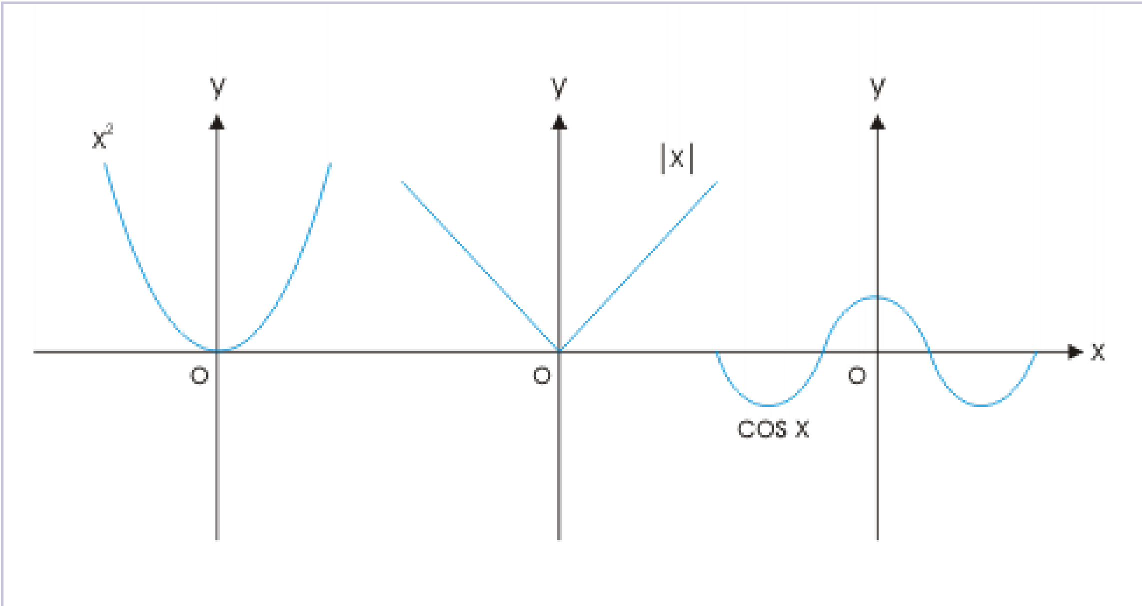

Figure 1: Examples of even functions.

It is important to see that if we rotate the curve by 180° about y-axis, then the appearance of the rotated curve is same as the original curve. We can state this alternatively as : if we rotate left hand side of the curve by 180° about y-axis, then we get the right hand curve and vice-versa.

2 Examples

2.1

Problem 1: Prove that the function f(x) is "even", if

$$f(x) = x\frac{a^x - 1}{a^x + 1}$$

Solution : For function being "even", we need to prove that :

$$f(-x) = f(x)$$

Here,

$$\Rightarrow f(-x) = -x\frac{a^{-x} - 1}{a^{-x} + 1} = -x\frac{\frac{1}{a^x} - 1}{\frac{1}{a^x} + 1}$$

$$\Rightarrow f(-x) = -x\frac{\frac{1-a^x}{a^x}}{\frac{1+a^x}{a^x}} = -x\frac{1 - a^x}{1 + a^x}$$

$$\Rightarrow f(-x) = x\frac{a^x - 1}{a^x + 1} = f(x)$$

2.2

Problem 2: If an even function "f" is defined on the interval (-5,5), then find the real values for which

$$f(x) = f\left(\frac{x+1}{x+2}\right)$$

Solution : It is given that function "f" is even. Hence, arguments of the functions on two sides are related either as

$$\Rightarrow x = \frac{x+1}{x+2}$$

or as :

$$\Rightarrow x = -\frac{x+1}{x+2}$$

From the first relation,

$$\Rightarrow x^2 + x - 1 = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{5}}{2}$$

From the second relation,

$$\Rightarrow x^2 + 3x + 1 = 0$$

$$\Rightarrow x = \frac{-3 \pm \sqrt{5}}{2}$$

We see that values are within the specified domain. Hence, all the four solutions satisfy the given equation.

3 Odd functions

The values of odd function at x=x and x=-x are equal in magnitude but opposite in sign.

Definition 2: Odd function

A function f(x) is said to be "odd" if for every "x", there exists "-x" in the domain of the function such that :

$$f(-x) = -f(x)$$

An odd function is symmetric about origin of the coordinate system. The plot in first quadrant has its mirror image (bilaterally inverted) in third quadrant. Similarly, the plot in second quadrant has its mirror image (bilaterally inverted) in fourth quadrant.

Some examples of odd functions are :x, x^3 and $\sin x$. In each case, we see that :

$$\Rightarrow f(-x) = -x = -f(x)$$

$$\Rightarrow f(-x) = (-x)^3 = -x^3 = -f(x)$$

$$\Rightarrow f(-x) = \sin(-x) = -\sin x = -f(x)$$

The upper curve of these functions is exactly same as the lower curve across x-axis.

Odd functions

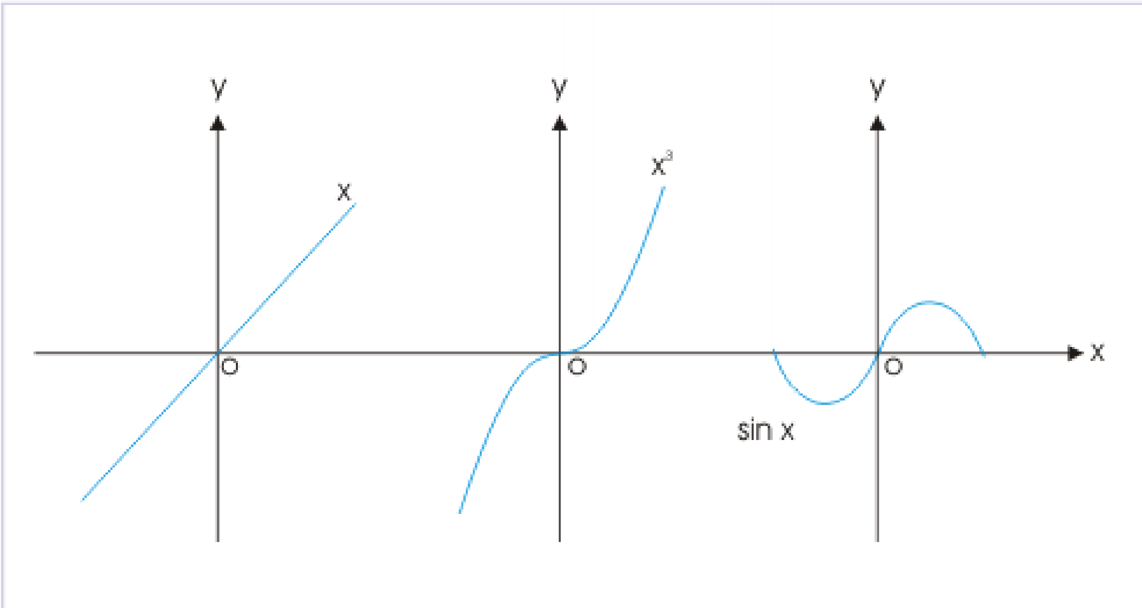

Figure 2: Examples of odd functions.

It is important to see that if we rotate the curve by 180° about origin, then the appearance of the rotated curve is same as the original curve. In other words, if we rotate right hand side of curve by 180° about origin, then we get left side of the curve. Further, it is interesting to note that we obtain left hand part of the plot of odd function in two steps : (i) drawing reflection (mirror image) of right hand plot about y-axis and (ii) drawing reflection (mirror image) of "reflection drawn in step 1" about x-axis.

Odd function plot

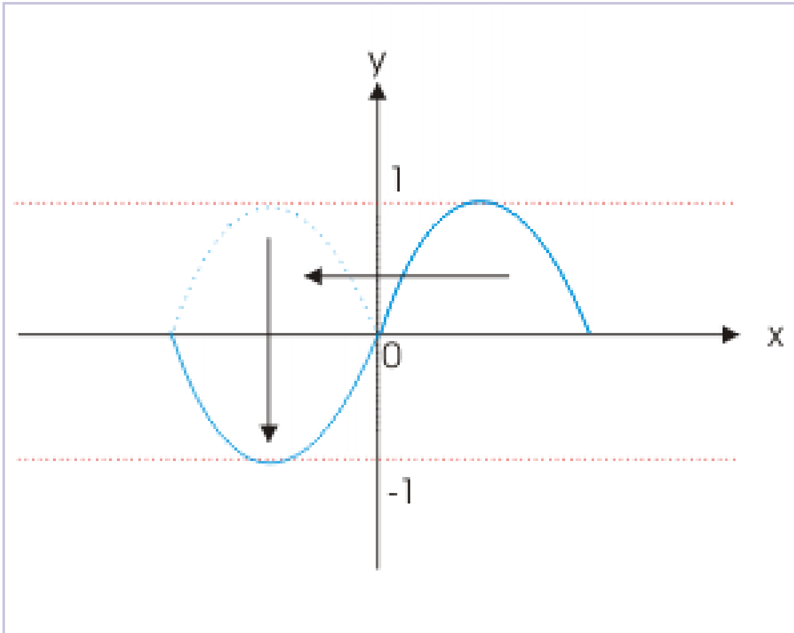

Figure 3: Odd function as two successive mirror images

4 Examples

4.1

Problem 3: Determine whether the function f(x) is "odd" function, where :

$$f\left(x\right)=\log_e\{x+\sqrt{(x^2+1)}\}$$

Solution : In order to determine the nature of function with respect to even or odd, we check for f(-x). Here,

$$\Rightarrow f\left(-x\right)=\log_e\left[-x+\sqrt{\{(-x)^2+1\}}\right]=\log_e\{-x+\sqrt{(x^2+1)}\}$$

The expression on the right hand side can not be explicitly interpreted whether it equals to f(x) or not. Therefore, we rationalize the expression of logarithmic function,

$$\Rightarrow f\left(-x\right)=\log_e\left[\frac{\{-x+\sqrt{(x^2+1)}\}X\{x+\sqrt{(x^2+1)}\}}{\{x+\sqrt{(x^2+1)}\}}\right]=\log_e\left[\frac{-x^2+x^2+1}{\{x+\sqrt{(x^2+1)}\}}\right]$$

$$\Rightarrow f\left(-x\right)=\log_e 1-\log_e\{x+\sqrt{(x^2+1)}\}=-\log_e\{x+\sqrt{(x^2+1)}\}=-f\left(x\right)$$

Hence, given function is an "odd" function.

4.2

Problem 4: Determine whether sinx + cosx is an even or odd function?

Solution : In order to check the nature of the function, we evaluate f(-x),

$$f(-x) = \sin(-x) + \cos(-x) = -\sin x + \cos x$$

The resulting function is neither equal to f(x) nor equal to “-f(x)”. Hence, the given function is neither an even nor an odd function.

5 Mathematical operations and nature of function

It is easy to find the nature of function resulting from mathematical operations, provided we know the nature of operand functions. As already discussed, we check for following possibilities :

- If f(-x) = f(x), then f(x) is even.
- If f(-x) = -f(x), then f(x) is odd.
- If above conditions are not met, then f(x) is neither even nor odd.

Based on above algorithm, we can determine the nature of resulting function. For example, let us determine the nature of "fog" function when “f” is an even and “g” is an odd function. By definition,

$$fog(-x) = f(g(-x))$$

But, “g” is an odd function. Hence,

$$\Rightarrow g(-x) = -g(x)$$

Combining two equations,

$$\Rightarrow fog(-x) = f(-g(x))$$

It is given that “f” is even function. Therefore, f(-x) = f(x). Hence,

$$\Rightarrow fog(-x) == f(-g(x)) = f(g(x)) = fog(x)$$

Therefore, resulting “fog” function is even function.

The nature of resulting function subsequent to various mathematical operations is tabulated here for reference :

f(x)	g(x)	f(x) ± g(x)	f(x) g(x)	f(x)/g(x), g(x)≠0	fog(x)
odd	odd	odd	even	even	odd
odd	even	Neither	odd	odd	even
even	even	even	even	even	even

We should emphasize here that we need not memorize this table. We can always carry out particular operation and determine whether a particular operation results in even, odd or neither of two function types. We shall work with a division operation here to illustrate the point. Let f(x) and g(x) be even and odd functions respectively. Let h(x) = f(x)/g(x). We now substitute “x” by “-x”,

$$\Rightarrow h(-x) = \frac{f(-x)}{g(-x)}$$

But f(x) is an even function. Hence, f(-x) = f(x). Further as g(x) is an odd function, g(-x) = - g(x).

$$\Rightarrow h\left(-x\right)=\frac{f\left(x\right)}{-g\left(x\right)}=-h\left(x\right)$$

Thus, the division, here, results in an odd function.

There is an useful parallel here to remember the results of multiplication and division operations. If we consider even as "plus (+)" and odd as "minus (-)", then the resulting function is same as that resulting from multiplication or division of plus and minus numbers. Product of even (plus) and odd (minus) is minus(odd). Product of odd (minus) and odd (minus) is plus (even). Similarly, division of odd (minus) by even (plus) is minus (odd) and so on.

Square of an even or odd function

The square of even or odd function is always an even function.

Properties of derivatives

1: If f(x) is an even differentiable function on R, then f'(x) is an odd function. In other words, if f(x) is an even function, then its first derivative with respect to "x" is an odd function.

2: If f(x) is an odd differentiable function on R, then f'(x) is an even function. In other words, if f(x) is an odd function, then its first derivative with respect to "x" is an even function.

6 Composition of a function

Every real function can be considered to be composed from addition of an even and an odd function. This composition is unique for every real function. We follow an algorithm to prove this as :

Let f(x) be a real function for x ∈ R. Then,

$$f\left(x\right)=f\left(x\right)+f\left(-x\right)\}-f\left(-x\right)\}$$

Rearranging,

$$f\left(x\right)=\frac{1}{2}\{f\left(x\right)+f\left(-x\right)\}+\frac{1}{2}\{f\left(x\right)-f\left(-x\right)\}=g\left(x\right)+h\left(x\right)$$

Now, we seek to determine the nature of functions "g(x)" and "h(x). For "g(x)", we have :

$$\Rightarrow g\left(-x\right)=\frac{1}{2}\left[f\left(-x\right)+f\{-\left(-x\right)\}\right]=\frac{1}{2}\{f\left(-x\right)+f\left(x\right)\}=g\left(x\right)$$

Thus, "g(x)" is an even function.

Similarly,

$$\Rightarrow h\left(-x\right)=\frac{1}{2}\left[f\left(-x\right)-f\{-\left(-x\right)\}\right]=\frac{1}{2}\{f\left(-x\right)-f\left(x\right)\}=-h\left(x\right)$$

Clearly, "h(x)" is an odd function. We, therefore, conclude that all real functions can be expressed as addition of even and odd functions.

7 Even and odd extensions of function

A function has three components – definition(rule), domain and range. What could be the meaning of extension of function? As a matter of fact, we can not extend these components. The concept of extending of function is actually not a general concept, but limited with respect to certain property of a function. Here, we shall consider few even and odd extensions. Idea is to complete a function defined in one half of its representation (x>=0) with other half such that resulting function is either even or odd function.

7.1 Even function

Let f(x) is defined in [0,a]. Then, even extension is defined as :

```
       |f(x);  0≤x≤a
g(x) = |
       | f(-x); -a≤x<0
```

The graphical interpretation of such extension is that graph of function f(x) is extended in other half which is mirror image of f(x) in y-axis i.e. image across y-axis.

7.2 Odd extension

Let f(x) is defined in [0,a]. Then, odd extension is defined as :

```
       | f(x);  0≤x≤a
g(x) = |
       | -f(x); -a≤x<0
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The graphical interpretation of such extension is that graph of function f(x) is extended in other half which is mirror image of f(x) in x-axis i.e. image across x-axis.

8 Exercises

Exercise 1 *(Solution on p. 9.)*

Determine whether f(x) is odd or even, when :

$$f\left(x\right) = e^{x} + e^{-x}$$

Exercise 2 *(Solution on p. 9.)*

Determine whether f(x) is odd or even, when :

$$f\left(x\right) = \frac{x}{e^{x} - 1} + \frac{x}{2}$$

Exercise 3 *(Solution on p. 9.)*

) How to check whether a pulse equation of the form

$$y = \frac{a}{\{(3x + 4t)^{2} + b\}}$$

is symmetric or asymmetric, here "a" and "b" are constants.

NOTE: Posted by Dr. R.K.Singhal through e-mail

Exercise 4 *(Solution on p. 9.)*

Determine whether f(x) is odd or even, when :

$$f\left(x\right) = x^{2}\cos x - |\sin x|$$

Exercise 5 *(Solution on p. 9.)*

Determine whether f(x) is odd or even, when :

$$f\left(x\right) = xe^{-x^{2}\tan^{2}x}$$

Solutions to Exercises in this Module

Solution to Exercise 1 (p. 8)

The function "f(x)" consists of exponential terms. Here,

$$\Rightarrow f\left(-x\right) = e^{-x} + e^{-(-x)} = e^{-x} + e^{x} = e^{x} + e^{-x} = f\left(x\right)$$

Hence, given function is even function.

Solution to Exercise 2 (p. 8)

The function "f(x)" consists of exponential terms. In order to check polarity, we determine f(-x) :

$$f\left(-x\right) = -\frac{x}{e^{-x}-1} + \frac{-x}{2} = -\frac{x}{1/e^{x}-1} - \frac{x}{2}$$

$$\Rightarrow f\left(-x\right) = -\frac{xe^{x}}{1-e^{x}} - \frac{x}{2}$$

We observe here that it might be tedious to reduce the expression to either "f(x)" or "-f(x)". However, if we evaluate f(x) – f(-x), then the resulting expression can be easily reduced to simpler form.

$$f\left(x\right) - f\left(-x\right) = \frac{x}{e^{x}-1} + \frac{x}{2} + \frac{xe^{x}}{1-e^{x}} + \frac{x}{2}$$

$$\Rightarrow f\left(x\right) - f\left(-x\right) = \frac{x}{e^{x}-1} - \frac{xe^{x}}{e^{x}-1} + x = \frac{x\left(1-e^{x}\right)}{e^{x}-1} + x = 0$$

Hence,

$$f\left(x\right) = f\left(-x\right)$$

It means that given function is an even function.

Solution to Exercise 3 (p. 8)

The pulse function has two independent variables "x" and "t". The function needs to be even for being symmetric about y-axis at a given instant, say t =0.

We check the nature of function at t = 0.

$$\Rightarrow y = \frac{a}{\left(9x^{2}+b\right)}$$

$$\Rightarrow f\left(-x\right) = \frac{a}{\left\{9(-x)^{2}+b\right\}} = \frac{a}{\left(9x^{2}+b\right)} = f\left(x\right)$$

Thus, we conclude that given pulse function is symmetric.

Solution to Exercise 4 (p. 8)

The "f(x)" function consists of trigonometric and modulus functions. Here,

$$\Rightarrow f\left(-x\right) = (-x)^{2}\cos\left(-x\right) - \left|\sin\left(-x\right)\right|$$

We know that :

$$(-x)^{2} = x^{2}; \quad \cos\left(-x\right) = \cos x; \quad \left|\sin\left(-x\right)\right| = \left|-\sin x\right| = \left|\sin x\right|$$

Putting these values in the expression of f(-x), we have :

$$\Rightarrow f\left(-x\right) = (-x)^{2}\cos\left(-x\right) - \left|\sin\left(-x\right)\right| = x^{2}\cos x - \left|\sin x\right| = f\left(x\right)$$

Hence, given function is an even function.

Solution to Exercise 5 (p. 8)

The "f(x)" function consists of exponential terms having trigonometric function in the exponent. Here,

$$\Rightarrow f\left(-x\right) = \left(-x\right) e^{-\{(-x)^2 \tan^2 (-x)\}}$$

We know that :

$$\left(-x\right)^2 = x^2; \quad \tan^2\left(-x\right) = \left(-\tan x\right)^2 = \tan^2 x$$

$$\Rightarrow f\left(-x\right) = \left(-x\right) e^{-\{(-x)^2 \tan^2 (-x)\}} = -x e^{-x^2 \tan^2 x} = -f\left(x\right)$$

Hence, given function is an odd function.