Precalculus

An Investigation of Functions

Edition 1.3

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Rational Function

A **rational function** is a function that can be written as the ratio of two polynomials, *P(x)* and *Q(x)*.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_p x^p}{b_0 + b_1 x + b_2 x^2 + \cdots + b_q x^q}$$

Example 3

A large mixing tank currently contains 100 gallons of water, into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after *t* minutes.

Notice that the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

$water = 100 + 10t$

$sugar = 5 + 1t$

The concentration, *C*, will be the ratio of pounds of sugar to gallons of water

$$C(t) = \frac{5+t}{100+10t}$$

Finding Asymptotes and Intercepts

Given a rational function, as part of investigating the short run behavior we are interested in finding any vertical and horizontal asymptotes, as well as finding any vertical or horizontal intercepts, as we have done in the past.

To find vertical asymptotes, we notice that the vertical asymptotes in our examples occur when the denominator of the function is undefined. With one exception, a vertical asymptote will occur whenever the denominator is undefined.

Example 4

Find the vertical asymptotes of the function $k(x) = \dfrac{5+2x^2}{2-x-x^2}$

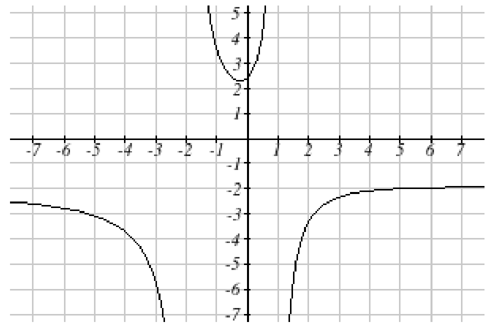
To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$2 - x - x^2 = 0$

$(2+x)(1-x) = 0$

$x = -2, 1$

This indicates two vertical asymptotes, which a look at a graph confirms.



The exception to this rule can occur when both the numerator and denominator of a rational function are zero at the same input.

Example 5

Find the vertical asymptotes of the function $k(x) = \dfrac{x-2}{x^2-4}$.

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

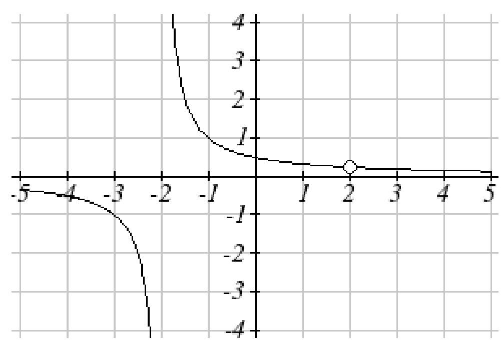
$$x^2 - 4 = 0$$

$$x^2 = 4$$

$$x = -2, 2$$

However, the numerator of this function is also equal to zero when $x = 2$. Because of this, the function will still be undefined at 2, since $\dfrac{0}{0}$ is undefined, but the graph will not have a vertical asymptote at $x = 2$.

The graph of this function will have the vertical asymptote at $x = -2$, but at $x = 2$ the graph will have a hole: a single point where the graph is not defined, indicated by an open circle.



Vertical Asymptotes and Holes of Rational Functions

The **vertical asymptotes** of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero.

A **hole** might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. In this case, factor the numerator and denominator and simplify; if the simplified expression still has a zero in the denominator at the original input the original function has a vertical asymptote at the input, otherwise it has a hole.

To find horizontal asymptotes, we are interested in the behavior of the function as the input grows large, so we consider long run behavior of the numerator and denominator separately. Recall that a polynomial's long run behavior will mirror that of the leading term. Likewise, a rational function's long run behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions.

There are three distinct outcomes when this analysis is done:

Case 1: The degree of the denominator > degree of the numerator
Example: $f(x) = \dfrac{3x+2}{x^2+4x-5}$
In this case, the long run behavior is $f(x) \approx \dfrac{3x}{x^2} = \dfrac{3}{x}$. This tells us that as the inputs grow large, this function will behave similarly to the function $g(x) = \dfrac{3}{x}$. As the inputs grow large, the outputs will approach zero, resulting in a horizontal asymptote at $y = 0$.
As $x \to \pm\infty$, $f(x) \to 0$

Case 2: The degree of the denominator < degree of the numerator
Example: $f(x) = \dfrac{3x^2+2}{x-5}$
In this case, the long run behavior is $f(x) \approx \dfrac{3x^2}{x} = 3x$. This tells us that as the inputs grow large, this function will behave similarly to the function $g(x) = 3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote.
As $x \to \pm\infty$, $f(x) \to \pm\infty$, respectively.

Ultimately, if the numerator is larger than the denominator, the long run behavior of the graph will mimic the behavior of the reduced long run behavior fraction. As another example if we had the function $f(x) = \dfrac{3x^5 - x^2}{x+3}$ with long run behavior $f(x) \approx \dfrac{3x^5}{x} = 3x^4$, the long run behavior of the graph would look similar to that of an even polynomial, and as $x \to \pm\infty$, $f(x) \to \infty$.

Case 3: The degree of the denominator = degree of the numerator
Example: $f(x) = \dfrac{3x^2+2}{x^2+4x-5}$

In this case, the long run behavior is $f(x) \approx \frac{3x^2}{x^2} = 3$. This tells us that as the inputs grow large, this function will behave like the function $g(x) = 3$, which is a horizontal line. As $x \to \pm\infty$, $f(x) \to 3$, resulting in a horizontal asymptote at $y = 3$.

Horizontal Asymptote of Rational Functions

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.
Degree of denominator > degree of numerator: Horizontal asymptote at $y = 0$
Degree of denominator < degree of numerator: No horizontal asymptote
Degree of denominator = degree of numerator: Horizontal asymptote at ratio of leading coefficients.

Example 6

In the sugar concentration problem from earlier, we created the equation
$C(t) = \frac{5+t}{100+10t}$.
Find the horizontal asymptote and interpret it in context of the scenario.

Both the numerator and denominator are linear (degree 1), so since the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is $10t$, with coefficient 10. The horizontal asymptote will be at the ratio of these values: As $t \to \infty$, $C(t) \to \frac{1}{10}$. This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the input gets large, the output values will approach 1/10. In context, this means that as more time goes by, the concentration of sugar in the tank will approach one tenth of a pound of sugar per gallon of water or 1/10 pounds per gallon.

Example 7

Find the horizontal and vertical asymptotes of the function
$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

First, note this function has no inputs that make both the numerator and denominator zero, so there are no potential holes. The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at x = 1, -2, and 5, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$.

Try it Now

3. Find the vertical and horizontal asymptotes of the function

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

Intercepts

As with all functions, a rational function will have a vertical intercept when the input is zero, if the function is defined at zero. It is possible for a rational function to not have a vertical intercept if the function is undefined at zero.

Likewise, a rational function will have horizontal intercepts at the inputs that cause the output to be zero (unless that input corresponds to a hole). It is possible there are no horizontal intercepts. Since a fraction is only equal to zero when the numerator is zero, horizontal intercepts will occur when the numerator of the rational function is equal to zero.

Example 8

Find the intercepts of $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

We can find the vertical intercept by evaluating the function at zero

$$f(0) = \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} = \frac{-6}{10} = -\frac{3}{5}$$

The horizontal intercepts will occur when the function is equal to zero:

$$0 = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$ This is zero when the numerator is zero

$$0 = (x-2)(x+3)$$

$$x = 2, -3$$

Try it Now

4. Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function and find the horizontal and vertical intercepts and the horizontal and vertical asymptotes.

3. Vertical asymptotes at x = 2 and x = -3; horizontal asymptote at y = 4

4. For the transformed reciprocal squared function, we find the rational form.

$$f(x) = \frac{1}{(x-3)^2} - 4 = \frac{1-4(x-3)^2}{(x-3)^2} = \frac{1-4(x^2-6x+9)}{(x-3)(x-3)} = \frac{-4x^2+24x-35}{x^2-6x+9}$$

Since the numerator is the same degree as the denominator we know that as $x \to \pm\infty, f(x) \to -4$. $y = -4$ is the horizontal asymptote. Next, we set the denominator equal to zero to find the vertical asymptote at x = 3, because as $x \to 3$, $f(x) \to \infty$. We set the numerator equal to 0 and find the horizontal intercepts are at (2.5,0) and (3.5,0), then we evaluate at 0 and the vertical intercept is at $\left(0, \frac{-35}{9}\right)$

5.
Horizontal asymptote at y = 1/2.
Vertical asymptotes are at x = 1, and x = 3.
Vertical intercept at (0, 4/3),
Horizontal intercepts (2, 0) and (-2, 0)
(-2, 0) is a double zero and the graph bounces off the axis at this point.
(2, 0) is a single zero and crosses the axis at this point.