Precalculus

An Investigation of Functions

Edition 1.3

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This is the fourth official version of Edition 1. It contains typo corrections and language clarification, but is page number and problem set number equivalent to the original Edition 1.

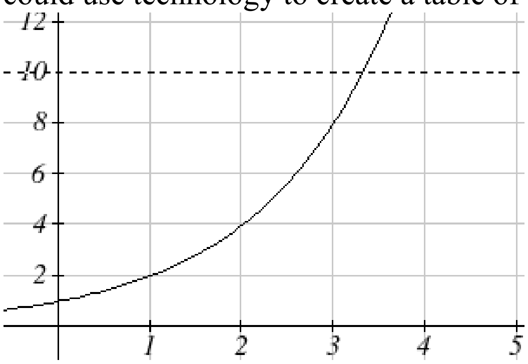
Section 4.3 Logarithmic Functions

A population of 50 flies is expected to double every week, leading to a function of the form $f(x) = 50(2)^x$, where *x* represents the number of weeks that have passed. When will this population reach 500? Trying to solve this problem leads to:

$500 = 50(2)^x$ Dividing both sides by 50 to isolate the exponential

$10 = 2^x$

While we have set up exponential models and used them to make predictions, you may have noticed that solving exponential equations has not yet been mentioned. The reason is simple: none of the algebraic tools discussed so far are sufficient to solve exponential equations. Consider the equation $2^x = 10$ above. We know that $2^3 = 8$ and $2^4 = 16$, so it is clear that *x* must be some value between 3 and 4 since $g(x) = 2^x$ is increasing. We could use technology to create a table of values or graph to better estimate the solution.



From the graph, we could better estimate the solution to be around 3.3. This result is still fairly unsatisfactory, and since the exponential function is one-to-one, it would be great to have an inverse function. None of the functions we have already discussed would serve as an inverse function and so we must introduce a new function, named **log** as the inverse of an exponential function. Since exponential functions have different bases, we will define corresponding logarithms of different bases as well.

Logarithm

The logarithm (base *b*) function, written $\log_b(x)$, is the inverse of the exponential function (base *b*), b^x.

Since the logarithm and exponential are inverses, it follows that:

Properties of Logs: Inverse Properties

$\log_b(b^x) = x$

$b^{\log_b x} = x$

Recall also from the definition of an inverse function that if $f(a) = c$, then $f^{-1}(c) = a$. Applying this to the exponential and logarithmic functions:

Logarithm Equivalent to an Exponential

The statement $b^a = c$ is equivalent to the statement $\log_b(c) = a$.

Alternatively, we could show this by starting with the exponential function $c = b^a$, then taking the log base *b* of both sides, giving $\log_b(c) = \log_b b^a$. Using the inverse property of logs we see that $\log_b(c) = a$.

Since log is a function, it is most correctly written as $\log_b(c)$, using parentheses to denote function evaluation, just as we would with *f(c)*. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written as $\log_b c$.

Example 1

Write these exponential equations as logarithmic equations:

$2^3 = 8$ $\quad$ $5^2 = 25$ $\quad$ $10^{-4} = \dfrac{1}{10000}$

$2^3 = 8$ is equivalent to $\log_2(8) = 3$

$5^2 = 25$ is equivalent to $\log_5(25) = 2$

$10^{-4} = \dfrac{1}{10000}$ is equivalent to $\log_{10}\left(\dfrac{1}{10000}\right) = -4$

Example 2

Write these logarithmic equations as exponential equations:

$\log_6\left(\sqrt{6}\right) = \dfrac{1}{2}$ $\quad$ $\log_3(9) = 2$

$\log_6\left(\sqrt{6}\right) = \dfrac{1}{2}$ is equivalent to $6^{1/2} = \sqrt{6}$

$\log_3(9) = 2$ is equivalent to $3^2 = 9$

Try it Now

Write the exponential equation $4^2 = 16$ as a logarithmic equation.

By establishing the relationship between exponential and logarithmic functions, we can now solve basic logarithmic and exponential equations by rewriting.

Example 3

Solve $\log_4(x) = 2$ for x.

By rewriting this expression as an exponential, $4^2 = x$, so $x = 16$

Example 4

Solve $2^x = 10$ for x.

By rewriting this expression as a logarithm, we get $x = \log_2(10)$

While this does define a solution, and an exact solution at that, you may find it somewhat unsatisfying since it is difficult to compare this expression to the decimal estimate we made earlier. Also, giving an exact expression for a solution is not always useful – often we really need a decimal approximation to the solution. Luckily, this is a task calculators and computers are quite adept at. Unluckily for us, most calculators and computers will only evaluate logarithms of two bases. Happily, this ends up not being a problem, as we'll see briefly.

Common and Natural Logarithms

The **common log** is the logarithm with base 10, and is typically written $\log(x)$.

The **natural log** is the logarithm with base e, and is typically written $\ln(x)$.

Example 5

Evaluate $\log(1000)$ using the definition of the common log.

To evaluate $\log(1000)$, we can say $x = \log(1000)$, then rewrite into exponential form using the common log base of 10.

$10^x = 1000$

From this, we might recognize that 1000 is the cube of 10, so $x = 3$.

We also can use the inverse property of logs to write $\log_{10}(10^3) = 3$

Values of the common log

number	number as exponential	log(number)
1000	10^3	3
100	10^2	2
10	10^1	1
1	10^0	0
0.1	10^{-1}	-1
0.01	10^{-2}	-2
0.001	10^{-3}	-3

Example 16

A company's sales can be modeled by the function $S(t) = 5000e^{0.12t}$, with t measured in years. Find the annual growth rate.

Noting that $1 + r = e^k$, then $r = e^{0.12} - 1 = 0.1275$, so the annual growth rate is 12.75%. The sales function could also be written in the form $S(t) = 5000(1 + 0.1275)^t$.

Important Topics of this Section

The Logarithmic function as the inverse of the exponential function
Writing logarithmic & exponential expressions
Properties of logs
- Inverse properties
- Exponential properties
- Change of base

Common log
Natural log
Solving exponential equations

Try it Now Answers

1. $\log_4(16) = 2 = \log_4 4^2 = 2\log_4 4$
2. 6
3. $-2\ln(x)$
4. $\frac{\ln(2)}{\ln(0.93)} \approx -9.5513$