Precalculus

An Investigation of Functions

Edition 1.3

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This is the fourth official version of Edition 1. It contains typo corrections and language clarification, but is page number and problem set number equivalent to the original Edition 1.

$$\frac{\ln 10}{\ln 2} \approx \frac{2.30259}{0.69315} \approx 3.3219$$

This finally allows us to answer our original question – the population of flies we discussed at the beginning of the section will take 3.32 weeks to grow to 500.

Example 11

Evaluate $\log_5(100)$ using the change of base formula.

We can rewrite this expression using any other base. If our calculators are able to evaluate the common logarithm, we could rewrite using the common log, base 10.

$$\log_5(100) = \frac{\log_{10} 100}{\log_{10} 5} \approx \frac{2}{0.69897} = 2.861$$

While we were able to solve the basic exponential equation $2^x = 10$ by rewriting in logarithmic form and then using the change of base formula to evaluate the logarithm, the proof of the change of base formula illuminates an alternative approach to solving exponential equations.

Solving exponential equations:

1. Isolate the exponential expressions when possible
2. Take the logarithm of both sides
3. Utilize the exponent property for logarithms to pull the variable out of the exponent
4. Use algebra to solve for the variable.

Example 12

Solve $2^x = 10$ for x.

Using this alternative approach, rather than rewrite this exponential into logarithmic form, we will take the logarithm of both sides of the equation. Since we often wish to evaluate the result to a decimal answer, we will usually utilize either the common log or natural log. For this example, we'll use the natural log:

$\ln\left(2^x\right) = \ln(10)$ Utilizing the exponent property for logs,

$x\ln\left(2\right) = \ln(10)$ Now dividing by ln(2),

$$x = \frac{\ln(10)}{\ln\left(2\right)} \approx 2.861$$

Notice that this result matches the result we found using the change of base formula.

Example 13

In the first section, we predicted the population (in billions) of India t years after 2008 by using the function $f(t) = 1.14(1+0.0134)^t$. If the population continues following this trend, when will the population reach 2 billion?

We need to solve for the t so that $f(t) = 2$

$2 = 1.14(1.0134)^t$ Divide by 1.14 to isolate the exponential expression

$\frac{2}{1.14} = 1.0134^t$ Take the logarithm of both sides of the equation

$\ln\left(\frac{2}{1.14}\right) = \ln\left(1.0134^t\right)$ Apply the exponent property on the right side

$\ln\left(\frac{2}{1.14}\right) = t\ln(1.0134)$ Divide both sides by ln(1.0134)

$$t = \frac{\ln\left(\frac{2}{1.14}\right)}{\ln(1.0134)} \approx 42.23 \text{ years}$$

If this growth rate continues, the model predicts the population of India will reach 2 billion about 42 years after 2008, or approximately in the year 2050.

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4. Solve $5(0.93)^x = 10$.

In addition to solving exponential equations, logarithmic expressions are common in many physical situations.

Example 14

In chemistry, pH is a measure of the acidity or basicity of a liquid. The pH is related to the concentration of hydrogen ions, $[H^+]$, measured in moles per liter, by the equation $pH = -\log\left(\left[H^+\right]\right)$.

If a liquid has concentration of 0.0001 moles per liber, determine the pH.
Determine the hydrogen ion concentration of a liquid with pH of 7.

To answer the first question, we evaluate the expression $-\log(0.0001)$. While we could use our calculators for this, we do not really need them here, since we can use the inverse property of logs:

$$-\log(0.0001) = -\log\left(10^{-4}\right) = -(-4) = 4$$

To answer the second question, we need to solve the equation $7 = -\log\left(\left[H^+\right]\right)$. Begin by isolating the logarithm on one side of the equation by multiplying both sides by -1:
$-7 = \log\left(\left[H^+\right]\right)$

Rewriting into exponential form yields the answer
$\left[H^+\right] = 10^{-7} = 0.0000001$ moles per liter.

Logarithms also provide us a mechanism for finding continuous growth models for exponential growth given two data points.

Example 15

A population grows from 100 to 130 in 2 weeks. Find the continuous growth rate.

Measuring t in weeks, we are looking for an equation $P(t) = ae^{rt}$ so that $P(0) = 100$ and $P(2) = 130$. Using the first pair of values,
$100 = ae^{r \cdot 0}$, so $a = 100$.

Using the second pair of values,

$130 = 100e^{r \cdot 2}$	Divide by 100
$\dfrac{130}{100} = e^{r2}$	Take the natural log of both sides
$\ln(1.3) = \ln\left(e^{r2}\right)$	Use the inverse property of logs
$\ln(1.3) = 2r$	
$r = \dfrac{\ln(1.3)}{2} \approx 0.1312$	

This population is growing at a continuous rate of 13.12% per week.

In general, we can relate the standard form of an exponential with the continuous growth form by noting (using k to represent the continuous growth rate to avoid the confusion of using r in two different ways in the same formula):

$$a(1+r)^x = ae^{kx}$$
$$(1+r)^x = e^{kx}$$
$$1+r = e^k$$

Using this, we see that it is always possible to convert from the continuous growth form of an exponential to the standard form and vice versa. Remember that the continuous growth rate k represents the nominal growth rate before accounting for the effects of continuous compounding, while r represents the actual percent increase in one time unit (one week, one year, etc.).

Example 16

A company's sales can be modeled by the function $S(t) = 5000e^{0.12t}$, with t measured in years. Find the annual growth rate.

Noting that $1 + r = e^k$, then $r = e^{0.12} - 1 = 0.1275$, so the annual growth rate is 12.75%. The sales function could also be written in the form $S(t) = 5000(1 + 0.1275)^t$.

Important Topics of this Section

The Logarithmic function as the inverse of the exponential function
Writing logarithmic & exponential expressions
Properties of logs
- Inverse properties
- Exponential properties
- Change of base

Common log
Natural log
Solving exponential equations

Try it Now Answers

1. $\log_4(16) = 2 = \log_4 4^2 = 2\log_4 4$
2. 6
3. $-2\ln(x)$
4. $\dfrac{\ln(2)}{\ln(0.93)} \approx -9.5513$