**Exercises on probabilities**

**Example 2.25**

In a manufacturing process, 10% of the parts contain visible surface flaws and 25% of the parts with surface flaws are functionally defective parts. However, only 5% of parts without surface flaws are functionally defective parts.

Make a graphical representation of the conditional probabilities

(conditional probability) The probability of a functionally defective part depends on the knowledge of the presence or absence of a surface flaw. If a part has a surface flaw, the probability of it being functionally defective is 0.25. If a part does not have a surface flaw, the probability of it being functionally defective is 0.05.

Let $A$ denote the event that a part is functionally defective

Let $B$ denote the event that the part has a surface flaw.

Then we are interested in the conditional probability $P(A|B) = 0.25$

$P(A|B') = 0.05$

The results are represented graphically below:

![Graphical representation of conditional probabilities](image)

**Example 2.28**

a) A history of 266 air samples has been classified on the basis of the presence of 2 rare molecules. Observations are represented in table below. Calculate the probability to observe an air sample with molecule 2 given that the air sample already contained molecule 1.

Calculate also the probability of observing molecule 1, molecule 2 and the probability to observe an air sample with molecule 1 given that the air sample already contained molecule 2.

<table>
<thead>
<tr>
<th>Molecule 1 present</th>
<th>Molecule 2 present</th>
</tr>
</thead>
<tbody>
<tr>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>212</td>
<td>24</td>
</tr>
<tr>
<td>yes</td>
<td>18</td>
</tr>
</tbody>
</table>

(conditional probability) Let $M_1$ denote the event that consists of all air samples in which rare molecule 1 is present, and $M_2$ denotes the event that consists of all air samples in which rare molecule 2 is present:
Calculate the probability of finding molecule 2 in the air samples if you know molecule 1 is present?

\[
P(\text{molecule2 present}|\text{molecule1 present}) = \frac{P(M_2|M_1)}{P(M_1)} = \frac{P(M_1 \cap M_2)}{P(M_1)} = \frac{12/266}{36/266}
\]

Calculate also \(P(M_1), P(M_2), P(M_1|M_2)\) and \(P(B|M_1)\)

\[
P(M_1) = \frac{36}{266}
\]
\[
P(M_2) = \frac{30}{266}
\]
\[
P(M_1|M_2) = \frac{12}{30}
\]
\[
P(M_2|M_1) = \frac{12}{36}
\]

\(P(M_1)\) and \(P(M_1|M_2)\) are the probabilities of the same event but are computed under two different states of knowledge. The results are also shown in a tree diagram. Remark that \(P(M_1|M_2) \neq P(M_1)\). The events \(M_1\) and \(M_2\) therefore are not mutually independent.

b) Example 2.33 Calculate the probability to observe an air sample with molecule 2 given that the air sample already contained molecule 1 and calculate the probability of observing an air sample containing molecule 2 (use the data below). Show that events \(A(M_1)\) and \(B(M_2)\) are independent.

\[
\begin{array}{c|c|c}
\text{Molecule 1 present} & \text{No} & \text{Yes} \\
\hline
\text{Molecule 2 present} & 230/266 & 36/266 \\
\hline
\text{No} & 212/230 & 18/230 \\
\hline
\text{Yes} & 12/36 & 24/36
\end{array}
\]

\(P(M_2|M_1)\) = \(24/36 = 1/3\)

\(P(M_2|M_1) = \frac{12/84}{36/84} = \frac{1}{3}\)

\(P(M_1|M_2) = \frac{12/84}{28/84} = \frac{3}{7}\)

\(P(M_1|M_2) = P(M_1) \) and \(P(M_2|M_1) = P(M_2) \Rightarrow \) independence

\[
P(M_1 \cap M_2) = P(M_1) * P(M_2) = P(M_1|M_2) * P(M_2) = 3/21 = 1/7 = 12/84
\]

In this example the knowledge that molecule 1 is present in a sample does not change the probability that molecule 2 is present. The event \(B\) consists of the same proportion of the total number of samples as the proportion of samples in \(A\). The 2 events are independent.
**Example 2.35**
Calculate the probability of observing 5 tosses in a sequence resulting in (head, head, head, tail, tail).

(independence) We assume that successful flips of a coin are independent and that the probability of a head on a flip is 0.5. Therefore the probability of throwing a head = probability of throwing a coin = 0.5: Because we have 5 independent throws P(S)= 0.5 X 0.5 X 0.5 X 0.5 X 0.5

**Example 2.39**
Because a new medical procedure has been shown to be effective in the early detection of an illness, a medical screening of the population is proposed. The probability that the test correctly identifies the illness as positive is 0.99, and the probability that the test correctly identifies someone without the illness as negative is 0.95. The incidence of the illness in the general population is 0.0001. You take the test and the result is positive. What is the probability that you have the illness.

(Bayes theorem) Let D denote the event that you have the illness, and let S denote the event that the test signals positive. The probability requested can be denoted as P(D|S), the probability that the test correctly signals someone without the illness is 0.95. Consequently, the probability of a positive test without the illness is P(S|D') = 0.05.

From Bayes Theorem,
P(D|S)= P(S|D) P(D)/[P(S|D)P(D) + P(S|D')P(D')]
=0.99(0.0001)/[0.99(0.0001)+0.05(1-0.0001)]

**Example 2-106**
The alignment between the magnetic tape and head in a magnetic tape storage system affects the performance of the system. Suppose 10% of the read operations are degraded by skewed alignments, 5% by off-center alignments, 1% by both skewness and off-center, and the remaining read operations are properly aligned. The probability of a read error is 0.01 from a skewed alignment 0.02 from an off-center alignment, 0.06 from both conditions, and 0.001 from a proper alignment. What is the probability of a read error?

Calculate the probability of the event read error? P(E) = ?

Suppose the event error read is represented by E
Suppose the event correct alignment is represented by C
Suppose the event skewed alignment is represented by S
Suppose the event off-center alignment is represented by O

P(S) = 0.1 and P(E|S)=0.01
P(O) = 0.05 and P(E|O)=0.02
P(O,S) = 0.01 and P(E|O,S)=0.06
P(C) = ? and P(E|C) = 0.001

P(E) = P(E|C)P(C) + P(E|S,O')P(S,O') + P(E|O,S')P(O,S') + P(E|S,O)P(S,O)
P(C) = 1- P(S U O) = 1 - [P(S) + P(O) – P(S,O) ] = 1 - [0.1+0.05-0.01] = 0.86
P(S,O') = P(S) - P(S,O) = 0.1 - 0.01 = 0.09
P(O,S') = P(O) - P(S,O) = 0.05 - 0.01 = 0.04

P(E,S) = P(E,S,O') + P(E,S,O)
P(E|S)P(S) = P(E|S,O') P(S,O') + P(E|S,O)P(S,O) Bayes
\[ P(E|S,O') = \frac{[P(E|S)P(S) - P(E|S,O)P(S,O)]}{P(S,O')} \]
\[ = \frac{[0.01\times0.1 - 0.06\times0.01]}{0.09} = 0.0044 \]

\[ P(E|O,S') = \frac{[P(E|S)P(S) - P(E|S,O)P(S,O)]}{P(O,S')} \]
\[ = \frac{[0.01\times0.1 - 0.06\times0.01]}{0.04} = 0.01 \]

\[ P(E) = P(E|C)P(C) + P(E|S,O')P(S,O') + P(E|O,S')P(O,S') + P(E|S,O)P(S,O) \]
\[ = 0.001\times0.86 + 0.0044\times0.09 + 0.01\times0.04 + 0.06\times0.01 \]
\[ = 0.0023 \]
\[ = 0.23\% \]

OR

\[ P(C) = 1 - P(S \cup O) = 1 - [P(S) + P(O) - P(S,O)] = 1 - [0.1 + 0.05 - 0.01] = 0.86 \]

\[ P(E) = P(E|C)P(C) + P(E|SUO)P(SUO) \]
\[ = P(E|C)P(C) + P(E|S)P(S) + P(E|O)P(O) - P(E|S,O)P(S,O) \]
\[ = 0.001\times0.86 + 0.01\times0.1 + 0.02\times0.05 - 0.01\times0.06 \]
\[ = 0.0023 \]